

DIMENSIONS OF FIBERS OF GENERIC CONTINUOUS MAPS

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ABSTRACT. In an earlier paper Buczolic, Elekes and the author described the Hausdorff dimension of the level sets of a generic real-valued continuous function (in the sense of Baire category) defined on a compact metric space K by introducing the notion of topological Hausdorff dimension. Later on, the author extended the theory for maps from K to \mathbb{R}^n . The main goal of this paper is to generalize the relevant results for topological and packing dimensions and to obtain new results for sufficiently homogeneous spaces K even in the case of Hausdorff dimension.

Let K be a compact metric space and denote by $C(K, \mathbb{R}^n)$ the Banach space of the continuous maps from K to \mathbb{R}^n . Let \dim_* be one of the topological dimension \dim_T , the Hausdorff dimension \dim_H , or the packing dimension \dim_P . Define

$$d_*^n(K) = \inf\{\dim_*(X \setminus F) : F \subset K \text{ is } \sigma\text{-compact with } \dim_T F < n\}.$$

We prove that $d_*^n(K)$ is the right notion to describe the dimensions of the fibers of a generic continuous map $f \in C(K, \mathbb{R}^n)$. In particular, we show that $\sup\{\dim_* f^{-1}(y) : y \in \mathbb{R}^n\} = d_*^n(K)$ provided that $\dim_T K \geq n$, otherwise every fiber is finite. Proving the above theorem for packing dimension requires entirely new ideas. Moreover, we show that the supremum is attained on the left hand side of the above equation.

Assume $\dim_T K \geq n$. If K is sufficiently homogeneous, then we can say much more. For example, we prove that $\dim_* f^{-1}(y) = d_*^n(K)$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \text{int } f(K)$ if and only if $d_*^n(U) = d_*^n(K)$ or $\dim_T U < n$ for all open sets $U \subset K$. This is new even if $n = 1$ and $\dim_* = \dim_H$. It is known that for a generic $f \in C(K, \mathbb{R}^n)$ the interior of $f(K)$ is not empty. We augment the above characterization by showing that $\dim_T \partial f(K) = \dim_H \partial f(K) = n - 1$ for a generic $f \in C(K, \mathbb{R}^n)$. In particular, almost every point of $f(K)$ is an interior point.

In order to obtain more precise results, we use the concept of generalized Hausdorff and packing measures, too.

1. INTRODUCTION

Let \dim_T , \dim_H , and \dim_P denote the topological, Hausdorff, and packing dimension, respectively. In this paper we adopt the convention $\dim \emptyset = -1$ for all dimensions. The following theorem is due to Kirchheim [15, Theorem 2].

Theorem 1.1 (Kirchheim). *Let $m, n \in \mathbb{N}^+$ with $m \geq n$. For a generic continuous map $f : [0, 1]^m \rightarrow \mathbb{R}^n$ (in the sense of Baire category) for all $y \in \text{int } f([0, 1]^m)$ we have*

$$\dim_H f^{-1}(y) = m - n.$$

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In order to generalize Kirchheim's theorem for real-valued functions defined on arbitrary compact metric spaces, Buczolich, Elekes, and the author introduced in [3] a new dimension for metric spaces, the *topological Hausdorff dimension*.

Recall the definition of (small inductive) topological dimension.

Definition 1.2. Set $\dim_T \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_T X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_T \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.$$

For more information on this concept see [7] or [13]. The topological Hausdorff dimension (introduced in [3]) is defined analogously to the topological dimension. However, it is not inductive, and it can attain non-integer values as well.

Definition 1.3. Set $\dim_{TH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{TH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.$$

The following more general definition is due to Darji and Elekes [5], the topological Hausdorff dimension corresponds to the case $n = 1$.

Definition 1.4. Let $\dim_{TH}^n \emptyset = -1$ for all $n \in \mathbb{N}$. For a non-empty metric space X set $\dim_{TH}^0 X = \dim_H X$. The *n th inductive topological Hausdorff dimension* is defined inductively as

$$\dim_{TH}^n X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_{TH}^{n-1} \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.$$

All above notions of dimension can attain the value ∞ as well, we use the convention $\infty - 1 = \infty$, hence $d = \infty$ is a member of the above sets. For more information see [3] and [2].

Here we repeat the definition of d_*^n extended to separable metric spaces as well.

Definition 1.5. Let $n \in \mathbb{N}^+$ and let X be a separable metric space. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . Then

$$d_*^n(X) = \inf\{\dim_*(X \setminus F) : F \subset X \text{ is an } F_\sigma \text{ set with } \dim_T F \leq n - 1\}.$$

As Hausdorff dimension admits G_δ hulls, [2, Theorem 4.4] yields the following.

Theorem 1.6. If $n \in \mathbb{N}^+$ and X is a separable metric space with $\dim_T X \geq n$ then

$$d_H^n(X) = \inf\{\dim_H(X \setminus A) : A \subset X \text{ with } \dim_T A < n\} = \dim_{TH}^n X - n.$$

Let \dim_* be one of \dim_T , \dim_H , or \dim_P . In Section 3 we prove some basic properties of d_*^n . We obtain a general upper bound for d_H^n and evaluate d_H^n and d_P^n for sets of the form $X \times [0, 1]^n$. We calculate $d_T^n(X)$ for separable metric spaces X .

Theorem 3.8. If $n \in \mathbb{N}^+$ and X is a separable metric space with $\dim_T X \geq n$ then

$$d_T^n(X) = \dim_T X - n.$$

Assume that a compact metric space K and $n \in \mathbb{N}^+$ are given, and let $C(K, \mathbb{R}^n)$ denote the space of continuous maps from K to \mathbb{R}^n equipped with the maximum norm. Since this is a complete metric space, we can use Baire category arguments.

If $\dim_T K < n$ then the fibers of a generic $f \in C(K, \mathbb{R}^n)$ are all finite, see Theorem 4.1.

From now on suppose that $\dim_T K \geq n$. By Theorem 1.6 the main theorem of [2] can be formalized as follows. The case $n = 1$ is due to Buczolich, Elekes, and the author, see [3, Theorem 7.12].

Theorem 1.7 (Balka). *Let $n \in \mathbb{N}^+$ and assume that K is a compact metric space with $\dim_T K \geq n$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

- (i) $\dim_H f^{-1}(y) \leq d_H^n(K)$ for all $y \in \mathbb{R}^n$,
- (ii) for every $d < d_H^n(K)$ there exists a non-empty open ball $U_{f,d} \subseteq \mathbb{R}^n$ such that $\dim_H f^{-1}(y) \geq d$ for every $y \in U_{f,d}$.

The main goal of this paper is to generalize of the above theorem for topological and packing dimensions as well. Our main result is the following.

Theorem 4.2 (Main Theorem, simplified version). *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

- (i) $\dim_* f^{-1}(y) \leq d_*^n(K)$ for all $y \in \mathbb{R}^n$,
- (ii) for every $d < d_*^n(K)$ there exists a non-empty open set $U_{f,d} \subset \mathbb{R}^n$ such that $\dim_* f^{-1}(y) \geq d$ for all $y \in U_{f,d}$.

Corollary 4.3. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . For a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\sup\{\dim_* f^{-1}(y) : y \in \mathbb{R}^n\} = d_*^n(K).$$

In Section 4 we prove Main Theorem (i). In Subsection 4.1 we show the Main Theorem for topological dimension, we simplify the proof of Theorem 1.7 by using the rich theory of topological dimension.

In Subsection 4.2 we prove the Main Theorem for packing dimension. We need to apply different methods, because the packing dimension, unlike the topological and Hausdorff dimensions, does not admit G_δ hulls. In fact, we prove a stronger result using the concept of generalized packing measure.

Theorem 4.12. *Let $n \in \mathbb{N}^+$ and let h be a left-continuous gauge function. Let K be a compact metric space such that $\dim_T U \geq n$ and $\mathcal{P}^h(U) = \infty$ for all non-empty open sets $U \subset K$. Then for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \text{int } f(K)$ we have*

$$\mathcal{P}^h(f^{-1}(y)) = \infty.$$

Moreover, the measures $\mathcal{P}^h|_{f^{-1}(y)}$ are not σ -finite.

The first part of the next theorem follows from a result of Kato [16, Theorem 4.6], while the second part is an easy corollary of Theorem 4.12.

Theorem 4.14. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

- (i) $\#f^{-1}(y) \leq n$ if $y \in \partial f(K)$,
- (ii) $\#f^{-1}(y) = 2^{\aleph_0}$ if $y \in \text{int } f(K)$.

If K is sufficiently homogeneous, then we can say much more. The following theorem is [2, Theorem 7.9]. The case $n = 1$ is due to Buczolic, Elekes, and the author, see [4, Theorem 3.6].

Theorem 1.8 (Balka). *Let $n \in \mathbb{N}^+$ and let K be a self-similar compact metric space with $\dim_T K \geq n$. For a generic $f \in C(K, \mathbb{R}^n)$ for any $y \in \text{int } f(K)$ we have*

$$\dim_H f^{-1}(y) = d_H^n(K).$$

If $d_*^n(K) = 0$ then the Main Theorem yields that $\dim_* f^{-1}(y) = d_*^n(K) = 0$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in f(K)$, so we may assume that $d_*^n(K) > 0$.

Now we characterize the compact sets K for which the statement of Theorem 1.8 holds. This is new even in the case of Hausdorff dimension and $n = 1$. For every $f \in C(K, \mathbb{R}^n)$ let

$$R_*(f) = \{y \in f(K) : \dim_* f^{-1}(y) = d_*^n(K)\}.$$

Theorem 5.2. *Let $n \in \mathbb{N}^+$ and assume that \dim_* is one of \dim_T , \dim_H , or \dim_P . Let K be a compact metric space with $d_*(K) > 0$. The following are equivalent:*

- (i) $R_*(f) = \text{int } f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (ii) $R_*(f)$ is dense in $\text{int } f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (iii) $d_*^n(U) = d_*^n(K)$ or $\dim_T U < n$ for all open sets $U \subset K$.

Since $\dim_T K \geq n$, we have $\text{int } f(K) \neq \emptyset$ for a generic $f \in C(K, \mathbb{R}^n)$, see Theorem 4.10. Thus the first statement of the above theorem is never vacuous.

We obtain the following corollary. The extra assumption is that each non-empty open subset of K has topological dimension at least n , which makes $\text{int } f(K)$ dense in $f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$.

Corollary 5.3. *Let $n \in \mathbb{N}^+$ and assume that \dim_* is one of \dim_T , \dim_H , or \dim_P . Let K be a compact metric space with $d_*^n(K) > 0$. The following are equivalent:*

- (1) $R_*(f) = \text{int } f(K)$ and $\text{int } f(K)$ is dense in $f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (2) $R_*(f)$ is dense in $f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (3) $d_*^n(U) = d_*^n(K)$ for all non-empty open sets $U \subset K$.

In particular, we obtain an extension of Kirchheim's theorem for topological and packing dimensions.

Corollary 5.4. *Let $m, n \in \mathbb{N}^+$ with $m \geq n$. For a generic $f \in C([0, 1]^m, \mathbb{R}^n)$ for all $y \in \text{int } f([0, 1]^m)$ we have*

$$\dim_T f^{-1}(y) = \dim_H f^{-1}(y) = m - n \quad \text{and} \quad \dim_P f^{-1}(y) = m.$$

Augmenting Theorem 1.1, Kirchheim [15, Theorem 2] proved the following.

Theorem 1.9 (Kirchheim). *Let $m \geq n$ be positive integers. Then for a generic $f \in C([0, 1]^m, \mathbb{R}^n)$ we have*

$$\dim_H \partial f(K) = n - 1.$$

In Section 6 we generalize this theorem by replacing $[0, 1]^m$ with an arbitrary compact metric space K satisfying $\dim_T K \geq n$.

Theorem 6.1. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\dim_T \partial f(K) = \dim_H \partial f(K) = n - 1.$$

Moreover, let h be a gauge function with $\lim_{r \rightarrow 0+} h(r)/r^{n-1} = 0$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have $\mathcal{H}^h(\partial f(K)) = 0$ and $\mathcal{H}^{n-1}(\partial f(K)) > 0$.

For each $f \in C(K, \mathbb{R}^n)$ let

$$S_*(f) = \{y \in f(K) : \dim_* f^{-1}(y) < d_*^n(K)\}.$$

Theorems 5.2 and 6.1 imply the following.

Corollary 6.3. *Let $n \in \mathbb{N}^+$ and assume that \dim_* is one of \dim_T , \dim_H , or \dim_P . Let K be a compact metric space with $d_*^n(K) > 0$. Exactly one of the following holds:*

- (a) $\dim_H S_*(f) = n - 1$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (b) $\text{int } S_*(f) \neq \emptyset$ for a generic $f \in C(K, \mathbb{R}^n)$.

Moreover, (a) is equivalent to the statements of Theorem 5.2.

In Section 7 we show that the supremum is attained in Corollary 4.3.

Theorem 7.2. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\max\{\dim_* f^{-1}(y) : y \in \mathbb{R}^n\} = d_*^n(K).$$

If $\dim_* = \dim_T$ and $\dim_T K < \infty$ then the Main Theorem yields more.

Corollary 7.1. *Assume that $n \in \mathbb{N}^+$ and K is a compact metric space satisfying $n \leq \dim_T K < \infty$. Then for a generic $f \in C(K, \mathbb{R}^n)$ there is a non-empty open set $U_f \subset \mathbb{R}^n$ such that for all $y \in U_f$ we have*

$$\dim_T f^{-1}(y) = d_T^n(K).$$

We show that otherwise Theorem 7.2 is sharp in general. For $n = 1$ the Hausdorff dimension part of the next theorem follows from [4, Theorem 4.5].

Theorem 7.6. *For each $n \in \mathbb{N}^+$ there is a compact set $K \subset \mathbb{R}^{n+1}$ such that for each $f \in C(K, \mathbb{R}^n)$ there is a $y_f \in \mathbb{R}^n$ such that*

- (i) $d_H^n(K) = 1$ and $d_P^n(K) = n + 1$,
- (ii) $\dim_H f^{-1}(y) < 1$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n \setminus \{y_f\}$,
- (iii) $\dim_P f^{-1}(y) < n + 1$ for every $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n \setminus \{y_f\}$.

Fact 7.7. *There is a compact metric space K such that $\dim_T K = \infty$, and for each $f \in C(K, \mathbb{R}^n)$ there is a $y_f \in \mathbb{R}^n$ such that $\dim_T f^{-1}(y) < \infty$ for all $y \in \mathbb{R}^n \setminus \{y_f\}$.*

2. PRELIMINARIES

Let (X, ρ) be a metric space, and let $A, B \subset X$ be arbitrary sets. We denote by $\text{cl } A$, $\text{int } A$, and ∂A the closure, interior, and boundary of A , respectively. The diameter of A is denoted by $\text{diam } A$. We use the convention $\text{diam } \emptyset = 0$. The distance of the sets A and B is defined by $\text{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. Let $B(x, r) = \{y \in X : \rho(x, y) \leq r\}$, and $U(x, r) = \{y \in X : \rho(x, y) < r\}$. We say that A is a *regular closed* set if $A = \text{cl}(\text{int } A)$. The set A is a *regular closed* set iff there is an open set $U \subset X$ such that $A = \text{cl } U$.

Fact 2.1. *Let X_0 be a metric space and let $k \in \mathbb{N}^+$. Let $X_k \subset X_{k-1} \subset \cdots \subset X_0$ such that $X_k \neq \emptyset$ and X_i is a regular closed set in the relative topology of X_{i-1} for all $i \in \{1, \dots, k\}$. Then X_k contains a non-empty open subset of X_0 .*

Let X be a *complete* metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subset X$ is *meager* if it is a countable union of nowhere dense sets, and a set is of *second category* if it is not meager. A set is called *co-meager* if its complement is meager. By Baire's category theorem a set is co-meager iff it contains a dense G_δ set. We say that the *generic* element $x \in X$ has property \mathcal{P} if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. The set $A \subset X$ has the *Baire property* if $A = U \Delta M$ where U is

open and M is meager. If $A \cap U$ contains a set which is of second category and has the Baire property for all non-empty open sets $U \subset X$, then A is co-meager. A separable, complete metric space is called a *Polish space*. If X is a Polish space and $A \subset X$, then A is *analytic* if it is a continuous image of a Polish space. Let $\sigma(\mathbf{A})$ denote the σ -algebra generated by the analytic sets. Clearly $\sigma(\mathbf{A})$ is closed under taking continuous images. Every analytic set has the Baire property, so sets in $\sigma(\mathbf{A})$ have the Baire property, too. For more information see [14].

If $E \subset X \times Y$ then for $x \in X$ let $E_x = \{y \in Y : (x, y) \in E\}$, and for $y \in Y$ let $E^y = \{x \in X : (x, y) \in E\}$. Let $\mathbf{0}$ denote the origin of \mathbb{R}^n .

If X is a metric space and A, B are disjoint subsets of X then we say that $P \subset X$ is a *partition between A and B* if there are open sets U, V such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$, and $P = X \setminus (U \cup V)$.

Topological dimension admits G_δ hulls, see [7, Theorem 1.5.11].

Theorem 2.2 (Enlargement theorem). *Let X be a metric space with a separable subspace $Y \subset X$. There is a G_δ set $G \subset X$ such that $Y \subset G$ and $\dim_T G = \dim_T Y$.*

For the following theorem see [7, Theorem 1.5.3].

Theorem 2.3 (Countable stability for F_σ sets). *Let X be a separable metric space and let $X_i \subset X$ be F_σ subsets such that $X = \bigcup_{i=1}^\infty X_i$. Then*

$$\dim_T X = \sup_{i \geq 1} \dim_T X_i.$$

The next result is [7, Theorem 1.5.10].

Theorem 2.4 (Addition theorem). *If X, Y are separable subspaces of a metric space then*

$$\dim_T(X \cup Y) \leq \dim_T X + \dim_T Y + 1.$$

See [7, Theorem 1.5.13] for the following.

Theorem 2.5 (Separation theorem). *Let X be a metric space with a separable subspace $Y \subset X$ and let $n \in \mathbb{N}^+$. If $\dim_T Y \leq n$, then for every pair (A, B) of disjoint closed subsets of X there is a partition P between A and B such that $\dim_T(P \cap Y) \leq n - 1$.*

The following theorem is [7, Theorem 1.5.8].

Theorem 2.6 (Decomposition theorem). *Let X be a separable metric space and let $n \in \mathbb{N}^+$. Then the following statements are equivalent:*

- (i) $\dim_T X \leq n$;
- (ii) $X = Z_1 \cup \dots \cup Z_{n+1}$ such that $\dim_T Z_i \leq 0$ for all $i \in \{1, \dots, n+1\}$.

For the next theorem see [7, Theorem 1.7.9].

Theorem 2.7 (Theorem on partitions). *Let X be a separable metric space and let $n \in \mathbb{N}^+$. Then the following statements are equivalent:*

- (i) $\dim_T X \leq n$;
- (ii) for every sequence $(A_1, B_1), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there are partitions P_i between A_i and B_i such that $\bigcap_{i=1}^{n+1} P_i = \emptyset$.

Let X be a metric space and let $A \subset X$. We use the convention $0^0 = 0$, $\inf \emptyset = \infty$, and $\sup \emptyset = 0$. We say that $h : [0, \infty) \rightarrow [0, \infty)$ is a *gauge function* if $h(0) = 0$ and h is non-decreasing. We define the *h -Hausdorff measure* of A as

$$\mathcal{H}^h(A) = \lim_{\delta \rightarrow 0+} \mathcal{H}_\delta^h(A), \text{ where}$$

$$\mathcal{H}_\delta^h(A) = \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam } A_i) : A \subset \bigcup_{i=1}^{\infty} A_i, \forall i \text{ diam } A_i \leq \delta \right\}.$$

If $h(x) = x^s$ for some $s \geq 0$ then we use the notation $\mathcal{H}^h = \mathcal{H}^s$ and $\mathcal{H}_\delta^h = \mathcal{H}_\delta^s$. The *Hausdorff dimension* of a non-empty A is defined as

$$\dim_H A = \inf \{s \geq 0 : \mathcal{H}^s(A) = 0\}.$$

The regularity of \mathcal{H}_δ^s easily implies that every set is contained in a G_δ set of the same Hausdorff dimension, and it is easy to see that countable stability holds.

For every $\delta > 0$, a δ -*packing* of A is a countable collection of disjoint balls $\{B(x_i, r_i)\}_{i \geq 1}$ with centres $x_i \in A$ and radii $0 \leq r_i \leq \delta$. For a gauge function h we define the *h -packing number* of A as

$$P_0^h(A) = \lim_{\delta \rightarrow 0+} P_\delta^h(A), \text{ where}$$

$$P_\delta^h(A) = \sup \left\{ \sum_{i=1}^{\infty} h(r_i) : \{B(x_i, r_i)\}_{i \geq 1} \text{ is a } \delta\text{-packing of } A \right\}.$$

As the countable subadditivity does not hold for P_0^h , we consider the following modification. The *h -packing measure* of A is defined as

$$\mathcal{P}^h(A) = \inf \left\{ \sum_{i=1}^{\infty} P_0^h(A_i) : A = \bigcup_{i=1}^{\infty} A_i \right\}.$$

If $h(x) = x^s$ for some $s \geq 0$ then we use the notation $\mathcal{P}^s = \mathcal{P}^h$. The *packing dimension* of a non-empty A is defined as

$$\dim_P A = \sup \{s \geq 0 : \mathcal{P}^s(A) > 0\}.$$

Countable stability holds for packing dimension, but it does not admit G_δ hulls.

Fact 2.8. *Let h be a left-continuous gauge function and let X be a metric space. Then $P_0^h(A) = P_0^h(\text{cl } A)$ for all $A \subset X$.*

The following lemma is [11, Lemma 4].

Lemma 2.9. *Let K be a compact metric space and let h be a left-continuous gauge function. If $P_0^h(U) = \infty$ for all non-empty open sets $U \subset K$, then $\mathcal{P}^h|_K$ is not σ -finite.*

For the following lemma see the proof of [20, Lemma 10.18]. For more special spaces X the lemma was originally proved in [19, Lemma 3.2] and [10, Lemma 4].

Lemma 2.10. *If X is a separable metric space satisfying $\dim_P X > s$, then there is a closed set $C \subset X$ such that $\dim_P(C \cap U) > s$ for all open sets U with $C \cap U \neq \emptyset$.*

For the following lemma see [12].

Lemma 2.11. *Let X be a non-empty metric space and let $n \in \mathbb{N}^+$. Then*

$$\dim_P(X \times [0, 1]^n) = \dim_P X + n.$$

For more information on Hausdorff and packing dimensions see [9] or [18].

The following lemma is standard, see e.g. [6, Lemma 3.8].

Lemma 2.12. *Let X, Y be complete metric spaces and let $R: X \rightarrow Y$ be a continuous open map. If $B \subset Y$ is co-meager, then $R^{-1}(B) \subset X$ is co-meager, too.*

As Tietze's extension theorem holds in \mathbb{R}^n , Lemma 2.12 implies the following.

Corollary 2.13. *Let $K_1 \subset K_2$ be compact metric spaces, and let $n \in \mathbb{N}^+$. Define*

$$R: C(K_2, \mathbb{R}^n) \rightarrow C(K_1, \mathbb{R}^n), \quad R(f) = f|_{K_1}.$$

If $\mathcal{F}_1 \subset C(K_1, \mathbb{R}^n)$ is co-meager, then $R^{-1}(\mathcal{F}_1) \subset C(K_2, \mathbb{R}^n)$ is co-meager, too.

For the following theorem see [16, Proposition 3.2].

Theorem 2.14 (Kato). *Let $n \in \mathbb{N}^+$ and let K be a compact metric space. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have $\dim_T f(K) \leq \dim_T K$.*

Lemma 2.15. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let U be the maximal open set $U \subset K$ with $\dim_T U < n$ and let $C = K \setminus U$. Then*

- (i) $\dim_T(C \cap U) \geq n$ for all open sets $U \subset K$ intersecting C ,
- (ii) $\text{int } f(C) = \text{int } f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$.

Proof. Note that clearly

$$U = \bigcup \{V \subset K : V \text{ is open and } \dim_T V < n\},$$

and the Lindelöf property of U and the countable stability of topological dimension for F_σ sets imply that $\dim_T U < n$.

First we prove (i). Assume to the contrary that there is an open set $V \subset K$ intersecting C such that $\dim_T(C \cap V) < n$. Clearly $\dim_T(V \setminus C) \leq \dim_T U < n$, so the countable stability of topological dimension for F_σ sets implies that $\dim_T V < n$. Then $V \subset U$ by definition, thus $C \cap V = \emptyset$. This is a contradiction, so (i) holds.

Now we show (ii). Let K_i be compact sets such that $U = \bigcup_{i=1}^\infty K_i$. Theorem 2.14 and Corollary 2.13 imply that for all $i \in \mathbb{N}^+$ a generic $f \in C(K, \mathbb{R}^n)$ satisfies $\dim_T f(K_i) \leq n - 1$. As a countable intersection of co-meager sets is co-meager and topological dimension is countably stable for closed sets, the set

$$\mathcal{F} = \{f \in C(K, \mathbb{R}^n) : \dim_T f(U) \leq n - 1\}$$

is co-meager in $C(K, \mathbb{R}^n)$. Assume $f \in \mathcal{F}$ and $W = \text{int } f(K)$, we need to prove that $W \subset f(C)$. Since $\dim_T f(U) < n$, the set $W \setminus f(U)$ is dense in W , and clearly $W \setminus f(U) \subset f(C)$. As $f(C)$ is compact, we have $W \subset f(C)$. This proves (ii). \square

3. SOME BASIC PROPERTIES OF d_*

Although our main examples for \dim_* will be \dim_T , \dim_H , and \dim_P , we might consider more general dimensions as well.

Definition 3.1. We say that \dim_* is a *notion of dimension*, if

- (i) $\dim_* \emptyset = -1$ and $\dim_* \{x\} = 0$ for each singleton x ,
- (ii) $\dim_* X \leq \dim_* Y$ for all separable metric spaces $X \subset Y$.

We can extend Definition 1.5 as follows.

Definition 3.2. Let $n \in \mathbb{N}^+$, let X be a separable metric space, and let \dim_* be a notion of dimension. Then

$$d_*^n(X) = \inf\{\dim_*(X \setminus F) : F \subset X \text{ is an } F_\sigma \text{ set with } \dim_T F \leq n-1\}.$$

Fact 3.3. *The infimum is attained in the above definition, that is,*

$$d_*^n(X) = \min\{\dim_*(X \setminus F) : F \subset X \text{ is an } F_\sigma \text{ set with } \dim_T F \leq n-1\}.$$

Proof. Assume that $F_i \subset X$ are F_σ -sets such that $\dim_T F_i \leq n-1$ for all $i \in \mathbb{N}^+$ and $\dim_*(X \setminus F_i) \rightarrow d_*^n(X)$ as $i \rightarrow \infty$. Let $F = \bigcup_{i=1}^\infty F_i$, then F is F_σ , too. As X is separable, the countable stability of topological dimension for F_σ sets implies that $\dim_T F \leq n-1$. The monotonicity of \dim_* yields that

$$\dim_*(X \setminus F) \leq \inf_{i \geq 1} \dim_*(X \setminus F_i) = d_*^n(X),$$

which completes the proof. \square

Fact 3.4. *Let X be a separable metric space, let \dim_* be a notion of dimension, and let $n \in \mathbb{N}^+$. Then*

$$d_*^n(X) = -1 \iff \dim_T X \leq n-1 \quad \text{and} \quad d_*^n(X) \geq 0 \iff \dim_T X \geq n.$$

The following fact easily follows from the definition of d_*^n .

Fact 3.5 (Monotonicity). *Let \dim_* be a notion of dimension and let $X \subset Y$ be separable metric spaces. Then $d_*^n(X) \leq d_*^n(Y)$ for all $n \in \mathbb{N}^+$.*

For the following theorem see [13, Theorem VII 2.] and [9].

Theorem 3.6. *For every separable metric space X we have*

$$\dim_T X \leq \dim_H X \leq \dim_P X.$$

The above theorem immediately implies the following.

Fact 3.7. *For every separable metric space X and $n \in \mathbb{N}^+$ we have*

$$d_T^n(X) \leq d_H^n(X) \leq d_P^n(X).$$

We are able to explicitly calculate $d_T^n(X)$.

Theorem 3.8. *Let $n \in \mathbb{N}^+$ and let X be a separable metric space with $\dim_T X \geq n$. Then*

$$d_T^n(X) = \dim_T X - n.$$

Proof. By Fact 3.3 there exists an F_σ set $F \subset X$ such that $\dim_T F \leq n-1$ and $d_T^n(X) = \dim_T(X \setminus F)$. By the addition theorem we have

$$\dim_T X \leq \dim_T(X \setminus F) + \dim_T F + 1 \leq d_T^n(X) + n.$$

Hence $d_T^n(X) \geq \dim_T X - n$.

For the other direction let $\dim_T X = d$. The decomposition theorem implies that X can be represented as a union $X = Z_1 \cup \dots \cup Z_{d+1}$, where $\dim_T Z_i \leq 0$ for all $1 \leq i \leq d+1$. As topological dimension admits G_δ hulls, there exists a G_δ set $G \subset X$ such that $Z_{n+1} \cup \dots \cup Z_{d+1} \subset G$ and $\dim_T G = \dim_T(Z_{n+1} \cup \dots \cup Z_{d+1})$. Then $F = X \setminus G$ is F_σ , and monotonicity and the addition theorem yield that

$$\dim_T F \leq \dim_T(Z_1 \cup \dots \cup Z_n) \leq n-1.$$

Similarly, we have

$$\dim_T(X \setminus F) = \dim_T G = \dim_T(Z_{n+1} \cup \dots \cup Z_{d+1}) \leq d - n = \dim_T X - n.$$

Therefore F witnesses that $d_T^n(X) \leq \dim_T X - n$. \square

Theorem 1.6 and [2, Theorem 3.5] yield an upper estimate for $d_H^n(X)$.

Theorem 3.9. *Let $n \in \mathbb{N}^+$ and let X be a separable metric space with $\dim_T X \geq n$. Then*

$$d_H^n(X) \leq \dim_H X - n.$$

By product of two metric spaces (X, d_X) and (Y, d_Y) we mean the ℓ^2 -product

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}.$$

It turns out that the upper estimate of Theorem 3.9 is sharp for product sets of the form $X \times [0, 1]^n$. For the following lemma see [2, Lemma 5.2].

Lemma 3.10. *Let X be a non-empty separable metric space and let $n \in \mathbb{N}^+$. Then*

$$d_H^n(X \times [0, 1]^n) = \dim_H X.$$

For $d_P^n(X)$ there is no non-trivial upper bound, $d_P^n(X) = \dim_P X$ is possible. We show that this is the case for product sets of the form $X \times [0, 1]^n$.

Lemma 3.11. *Let X be a non-empty Polish space and let $n \in \mathbb{N}^+$. Then*

$$d_P^n(X \times [0, 1]^n) = \dim_P(X \times [0, 1]^n) = \dim_P X + n.$$

Proof. Lemma 2.11 implies that $d_P^n(X \times [0, 1]^n) \leq \dim_P(X \times [0, 1]^n) = \dim_P X + n$.

For the other direction let $Z = X \times [0, 1]^n$ and let $d < \dim_P X$ be fixed, it is enough to prove that $d_P^n(Z) > d + n$. Let $F \subset Z$ be an F_σ set with $\dim_T F \leq n - 1$, it is enough to show that $P^{d+n}(Z \setminus F) > 0$. Let $\{A_i\}_{i \geq 1}$ be an arbitrary cover of $Z \setminus F$, it is enough to prove that $P_0^{d+n}(A_i) > 0$ for some i . By Lemma 2.10 there is a closed set $Y \subset X$ such that every non-empty relatively open set U in Y satisfies $\dim_P U > d$. Thus every non-empty relatively open set $V \subset Y \times [0, 1]^n$ satisfies $\dim_P V > d + n$, so $P_0^{d+n}(V) > 0$. Indeed, there is a relatively open set U in Y and a cube $Q \subset [0, 1]^n$ such that $U \times Q \subset V$, so applying Lemma 2.11 we obtain that

$$\dim_P V \geq \dim_P(U \times Q) = \dim_P U + n > d + n.$$

Clearly $\dim_T F \leq n - 1$ yields that F is meager in $\{y\} \times [0, 1]^n$ for all $y \in Y$, so F is meager in $Y \times [0, 1]^n$ by the Kuratowski-Ulam Theorem [14, Theorem 8.41]. Since $Y \times [0, 1]^n$ is complete, Baire's category theorem implies that for some i the set A_i is somewhere dense in $Y \times [0, 1]^n$, so there is a non-empty relatively open set $V \subset Y \times [0, 1]^n$ such that $V \subset \text{cl } A_i$. Therefore by Fact 2.8 we obtain

$$P_0^{d+n}(A_i) = P_0^{d+n}(\text{cl } A_i) \geq P_0^{d+n}(V) > 0,$$

and the proof is complete. \square

Theorem 3.12. *Let $n \in \mathbb{N}^+$ and let \dim_* be a notion of dimension. Assume that one of the following holds:*

- (i) \dim_* is countably stable,
- (ii) \dim_* is countably stable for closed sets and admits G_δ hulls.

Then d_^n is countably stable for closed sets.*

Proof. Let X be a separable metric space and assume that $X_i \subset X$ are closed such that $X = \bigcup_{i=1}^{\infty} X_i$, we have to prove that $d_*^n(X) = \sup_i d_*^n(X_i)$.

Monotonicity clearly implies $d_*^n(X) \geq \sup_i d_*^n(X_i)$, so it is enough to prove the other direction. By Fact 3.3 for each $i \in \mathbb{N}^+$ there exists an F_σ set $F_i \subset X_i$ such that $\dim_T F_i \leq n-1$ and $\dim_*(X_i \setminus F_i) = d_*^n(X_i)$. Let $F = \bigcup_{i=1}^{\infty} F_i$. As X_i are closed, F is F_σ in X . As F_i are F_σ in F as well, the countable stability of topological dimension for F_σ -sets implies that $\dim_T F \leq n-1$.

First assume that \dim_* is countably stable, then

$$d_*^n(X) \leq \dim_*(X \setminus F) = \sup_{i \geq 1} \dim_*(X_i \setminus F) = \sup_{i \geq 1} d_*^n(X_i).$$

Now assume that \dim_* is countably stable for closed sets and admits G_δ hulls. As the topological dimension admits G_δ hulls, there exists a G_δ set $G \subset X$ such that $F \subset G$ and $\dim_T G \leq n-1$. The countable stability of \dim_* for closed sets and monotonicity yield that

$$(3.1) \quad \dim_*(X \setminus G) = \sup_{i \geq 1} \dim_*(X_i \setminus G) \leq \sup_{i \geq 1} \dim_*(X_i \setminus F_i) = \sup_{i \geq 1} d_*^n(X_i).$$

Clearly \dim_* is countably stable for F_σ sets, too. Therefore there exists a G_δ set $H \subset X$ such that $X \setminus G \subset H$ and $\dim_* H = \dim_*(X \setminus G)$. Then $X \setminus H$ is F_σ and $\dim_T(X \setminus H) \leq \dim_T G \leq n-1$. Thus the definition of $d_*^n(X)$ and (3.1) imply that

$$d_*^n(X) \leq \dim_* H = \dim_*(X \setminus G) \leq \sup_{i \geq 1} d_*^n(X_i).$$

The proof is complete. \square

Corollary 3.13. *Let $n \in \mathbb{N}^+$ and let \dim_* be one of \dim_T , \dim_H , or \dim_P . Then d_*^n is countably stable for F_σ sets.*

4. THE MAIN THEOREM

The goal of this section is to describe the dimensions of the fibers of a generic $f \in C(K, \mathbb{R}^n)$ by proving our Main Theorem. Note that the case $\dim_T K < n$ is completed by the following theorem of Hurewicz [17, page 124].

Theorem 4.1 (Hurewicz). *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K < n$. Then $\#f^{-1}(y) \leq n$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n$.*

Consider the following technical version of the Main Theorem.

Theorem 4.2 (Main Theorem). *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . Then there exists a G_δ set $G \subset K$ with $\dim_* G = d_*^n(K)$ such that for a generic $f \in C(K, \mathbb{R}^n)$ we have*

- (i) $\#(f^{-1}(y) \setminus G) \leq n$ for all $y \in \mathbb{R}^n$, thus $\dim_* f^{-1}(y) \leq d_*^n(K)$ for all $y \in \mathbb{R}^n$,
- (ii) for every $d < d_*^n(K)$ there exists a non-empty open set $U_{f,d} \subset \mathbb{R}^n$ such that $\dim_* f^{-1}(y) \geq d$ for all $y \in U_{f,d}$.

Corollary 4.3. *Let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\sup\{\dim_* f^{-1}(y) : y \in \mathbb{R}^n\} = d_*^n(K).$$

Theorem 4.1 yields the following corollary, which states that a generic continuous map $f \in C(K, \mathbb{R}^n)$ is almost injective on any given F_σ set of topological dimension smaller than n .

Corollary 4.4. *Let K be a compact metric space and let $n \in \mathbb{N}^+$. Assume that $F \subset K$ is an F_σ set such that $\dim_T F < n$. Then $\#(f^{-1}(y) \cap F) \leq n$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n$.*

Proof. Choose compact sets K_i such that $F = \bigcup_{i=1}^\infty K_i$ and $K_i \subset K_{i+1}$ for all i . For all $i \in \mathbb{N}^+$ let

$$\mathcal{F}_i = \{f \in C(K_i, \mathbb{R}^n) : \#f^{-1}(y) \leq n \text{ for all } y \in \mathbb{R}^n\},$$

and define

$$R_i : C(K, \mathbb{R}^n) \rightarrow C(K_i, \mathbb{R}^n), \quad R_i(f) = f|_{K_i}.$$

Finally, let us define

$$\mathcal{F} = \bigcap_{i=1}^\infty R_i^{-1}(\mathcal{F}_i) \subset C(K, \mathbb{R}^n).$$

As $\dim_T K_i \leq \dim_T F \leq n-1$, Theorem 4.1 yields that the sets $\mathcal{F}_i \subset C(K_i, \mathbb{R}^n)$ are co-meager. Corollary 2.13 implies that $R_i^{-1}(\mathcal{F}_i) \subset C(K, \mathbb{R}^n)$ are co-meager, too. As a countable intersection of co-meager sets $\mathcal{F} \subset C(K, \mathbb{R}^n)$ is also co-meager. Fix $f \in \mathcal{F}$, we have $\#(f^{-1}(y) \cap K_i) \leq n$ for all $y \in \mathbb{R}^n$ and $i \in \mathbb{N}^+$. Thus $\bigcup_{i=1}^\infty K_i = F$ and $K_i \subset K_{i+1}$ imply that $\#(f^{-1}(y) \cap F) \leq n$ for all $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$. \square

Therefore we may remove an F_σ set F of topological dimension smaller than n from K , and the dimension of $K \setminus F$ estimates the dimension of the fibers of a generic continuous map from above. This yields the first half of the Main Theorem.

Proof of Main Theorem (i). By Fact 3.3 there is an F_σ set $F \subset K$ such that $\dim_T F \leq n-1$ and $d_*^n(K) = \dim_*(K \setminus F)$. Let $G = K \setminus F$, then $\dim_* G = d_*^n(K)$ and Corollary 4.4 implies that for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n$ we have $\#(f^{-1}(y) \setminus G) = \#(f^{-1}(y) \cap F) \leq n$. As $\dim_T K \geq n$ yields $G \neq \emptyset$, we have $\dim_* f^{-1}(y) \leq \dim_* G = d_*^n(K)$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n$. \square

The second part of the Main Theorem is much deeper, it states that the easy upper bound of the first part is always sharp. In case of Hausdorff dimension this is the main result of [2], see Theorem 1.7. In the following two subsections we prove the Main Theorem in case of topological and packing dimensions.

4.1. Topological dimension. In this subsection we prove the second part of the Main Theorem for topological dimension. The proof will be based on the proof of [2, Theorem 6.12], but it will be much shorter. This is because we can use the powerful theory of topological dimension, which in particular implied Theorem 3.8. The goal of this subsection is to prove the following.

Theorem 4.5. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then for a generic $f \in C(K, \mathbb{R}^n)$ for all $d < d_T^n(K)$ there is a non-empty open set $U_{f,d} \subset \mathbb{R}^n$ such that for all $y \in U_{f,d}$ we have $\dim_T f^{-1}(y) \geq d$.*

First we need some preparation. Assume that K is fixed with $\dim_T K \geq n$.

Definition 4.6. We say that $f \in C(K, \mathbb{R}^n)$ is *d-level narrow* if there is a dense set $S_f \subset \mathbb{R}^n$ such that $\dim_T f^{-1}(y) \leq d$ for all $y \in S_f$. Let $\mathcal{N}^n(d)$ be the set of *d-level narrow maps*. Let

$$N_n = \{d : \mathcal{N}^n(d) \text{ is somewhere dense in } C(K, \mathbb{R}^n)\},$$

$$D_n = \{d : \exists \text{ an } F_\sigma \text{ set } F \subset K \text{ with } \dim_T F \leq n-1 \text{ and } \dim_T(K \setminus F) \leq d\}.$$

We adopt the convention that $\infty \in N_n, D_n$.

Theorem 4.7. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then*

$$d_T^n(K) = \min N_n.$$

The following two lemmas and Fact 3.3 imply Theorem 4.7.

Lemma 4.8. $D_n \subset N_n$.

Proof. Assume that $d \in D_n$ and $d < \infty$. The first part of the Main Theorem yields that for a generic $f \in C(K, \mathbb{R}^n)$ we have $\dim_T f^{-1}(y) \leq d$ for all $y \in \mathbb{R}^n$. Therefore $\mathcal{N}^n(d)$ is co-meager, thus everywhere dense in $C(K, \mathbb{R}^n)$. Hence $d \in N_n$. \square

The proof of the following lemma is based on the proof of [2, Lemma 6.11].

Lemma 4.9. $N_n \subset D_n$.

Proof. Assume that $d \in N_n$ is a natural number, by Fact 3.3 it is enough to prove that $d_T^n(K) \leq d$. As $d_T^n(K) = \dim_T K - n$ by Theorem 3.8, it is enough to show that $\dim_T K \leq n + d$. Fix $f \in C(K, \mathbb{R}^n)$ and $\varepsilon > 0$ such that $\mathcal{N}^n(d)$ is dense in $B(f, \varepsilon)$. The uniform continuity of f implies that there is a $\delta > 0$ such that if $A \subset K$ with $\text{diam } A \leq \delta$ then $\text{diam } f(A) < \varepsilon/n$. The stability of topological dimension implies that there is a compact set $C \subset K$ such that $\text{diam } C \leq \delta$ and $\dim_T C = \dim_T K$, so we may assume that $\text{diam } K \leq \delta$.

Let $(A_1, B_1), \dots, (A_{n+d+1}, B_{n+d+1})$ be arbitrary pairs of disjoint closed subsets of K , then by the theorem on partitions it is enough to prove that there exist partitions $P_i \subset K$ between A_i and B_i such that $\bigcap_{i=1}^{n+d+1} P_i = \emptyset$. First we show that it is enough to find partitions $P_i \subset K$ between A_i and B_i for $1 \leq i \leq n$ such that

$$(4.1) \quad \dim_T \left(\bigcap_{i=1}^n P_i \right) \leq d.$$

Indeed, assuming (4.1) and applying the separation theorem $(d+1)$ -times yield that for all $1 \leq j \leq d+1$ there are partitions P_{n+j} between A_{n+j} and B_{n+j} with

$$\dim_T \left(\bigcap_{i=1}^{n+j} P_i \right) \leq d - j.$$

Therefore $\dim_T \left(\bigcap_{i=1}^{n+d+1} P_i \right) = -1$, so $\bigcap_{i=1}^{n+d+1} P_i = \emptyset$.

Finally, we prove (4.1). Let $f_1, \dots, f_n \in C(K, \mathbb{R})$ be such that $f = (f_1, \dots, f_n)$ and observe that we may construct for all $i \in \{1, \dots, n\}$ functions $g_i \in C(K, \mathbb{R})$ such that

- (i) $\max g_i(A_i) < \min g_i(B_i)$;
- (ii) $g_i \in B(f_i, \varepsilon/n)$;
- (iii) The map $g = (g_1, \dots, g_n) \in C(K, \mathbb{R}^n)$ satisfies $g \in \mathcal{N}^n(d)$.

Indeed, as $\text{diam } f_i(K) \leq \text{diam } f(K) < \varepsilon/n$, for every $i \in \{1, \dots, n\}$ we can define g_i first on $A_i \cup B_i$ and then we can extend it to K by Tietze's extension theorem such that (i) and (ii) hold. Property (ii) implies that $g = (g_1, \dots, g_n) \in B(f, \varepsilon)$. As $g \in B(f, \varepsilon)$ and $\mathcal{N}^n(d)$ is dense in $B(f, \varepsilon)$, we may assume that $g \in \mathcal{N}^n(d)$, so (iii) holds.

As $g \in \mathcal{N}^n(d)$, there is a dense set $S_g \subset \mathbb{R}^n$ such that $\dim_T g^{-1}(s) \leq d$ for all $s \in S_g$. We can choose $s = (s_1, \dots, s_n) \in S_g$ such that for every $i \in \{1, \dots, n\}$ its i th coordinate s_i satisfies $\max g_i(A_i) < s_i < \min g_i(B_i)$. For all $i \in \{1, \dots, n\}$ let

$$S_i = \{(y_1, \dots, y_n) \in g(K) : y_i = s_i\}.$$

Then (i) implies that S_i is a partition between $g(A_i)$ and $g(B_i)$ in $g(K)$ for every $i \in \{1, \dots, n\}$. For all i define $P_i = g^{-1}(S_i)$. Then P_i is a partition between A_i and B_i such that

$$\bigcap_{i=1}^n P_i = \bigcap_{i=1}^n g^{-1}(S_i) = g^{-1}\left(\bigcap_{i=1}^n S_i\right) = g^{-1}(s).$$

Therefore $s \in S_g$ implies that

$$\dim_T \left(\bigcap_{i=1}^n P_i \right) \leq \dim_T g^{-1}(s) \leq d.$$

Thus (4.1) holds, and the proof is complete. \square

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. Theorems 3.8 and 4.7 imply that $d_T^n(K) = \min N_n$. Choose a sequence $d_k \nearrow d_T^n(K)$, then $\mathcal{N}^n(d_k)$ is nowhere dense by the definition of N_n . Thus for every $f \in C(K, \mathbb{R}^n) \setminus \mathcal{N}^n(d_k)$ there exists a non-empty open set $U_{f,d_k} \subset \mathbb{R}^n$ such that $\dim_T f^{-1}(y) \geq d_k$ for every $y \in U_{f,d_k}$. But then the theorem holds for every $f \in C(K, \mathbb{R}^n) \setminus (\bigcup_{k=1}^\infty \mathcal{N}^n(d_k))$, and this latter set is co-meager. \square

Theorem 4.5 clearly implies the following known result, which will be useful later.

Theorem 4.10. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then $\text{int } f(K) \neq \emptyset$ for a generic $f \in C(K, \mathbb{R}^n)$.*

In fact, the origins of the above theorem date back to Hurewicz-Wallmann [13], and Alexandroff [1]. See the remark after [6, Theorem 1.7] for more explanation.

4.2. Packing dimension. The main goal of this subsection is to prove the second part of the Main Theorem for packing dimension. Topological and Hausdorff dimensions admit G_δ hulls, which was a crucial ingredient of the proof of the Main Theorem for these dimensions. Since packing dimension does not admit G_δ hulls, we need to come up with different arguments to prove the following.

Theorem 4.11. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. For a generic $f \in C(K, \mathbb{R}^n)$ for every $d < d_P^n(K)$ there exists a non-empty open set $U_{f,d} \subset \mathbb{R}^n$ such that $\dim_P f^{-1}(y) \geq d$ for all $y \in U_{f,d}$.*

Actually, we prove a stronger statement by using the concept of generalized packing measure.

Theorem 4.12. *Let $n \in \mathbb{N}^+$ and let h be a left-continuous gauge function. Let K be a compact metric space such that $\dim_T K \geq n$ and $\mathcal{P}^h(U) = \infty$ for all non-empty open sets $U \subset K$. Then for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \text{int } f(K)$ we have*

$$\mathcal{P}^h(f^{-1}(y)) = \infty.$$

Moreover, the measures $\mathcal{P}^h|_{f^{-1}(y)}$ are not σ -finite.

Remark 4.13. Haase [11] proved that if \mathcal{P}^h is not σ -finite on K , then K contains 2^{\aleph_0} disjoint compact subsets with non- σ -finite \mathcal{P}^h measure. Theorems 4.12 and 4.10 imply that if K everywhere at least n -dimensional, then a generic $f \in C(K, \mathbb{R}^n)$ provides 2^{\aleph_0} disjoint fibers of non- σ -finite \mathcal{P}^h measure.

The following theorem characterizes the infinite fibers of a generic $f \in C(K, \mathbb{R}^n)$. Part (i) follows from a result of Kato [16, Theorem 4.6]. Part (ii) might be known as well, but we could not find it in the literature. Instead, we show that it easily follows from Theorem 4.12.

Theorem 4.14. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

- (i) $\#f^{-1}(y) \leq n$ if $y \in \partial f(K)$,
- (ii) $\#f^{-1}(y) = 2^{\aleph_0}$ if $y \in \text{int } f(K)$.

Proof. We only prove (ii). By Lemma 2.15 and Corollary 2.13 we may assume that $\dim_T U \geq n$ for all non-empty open sets $U \subset K$. Let h be the gauge function such that $h(0) = 0$ and $h(x) = 1$ if $x > 0$. Applying Theorem 4.12 for h yields that for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \text{int } f(K)$ the measures $\mathcal{P}^h|_{f^{-1}(y)}$ are not σ -finite. As $\mathcal{P}^h = \mathcal{P}^0$ is the counting measure, each $f^{-1}(y)$ is uncountable. Then $\#f^{-1}(y) = 2^{\aleph_0}$ by [14, Corollary 6.5], and (ii) follows. \square

Now we prove Theorem 4.11 based on Theorem 4.12. We need the following lemma.

Lemma 4.15. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. If $d < d_P^n(K)$ then there exists a compact set $C \subset K$ such that for every open set U intersecting C we have $\dim_T(C \cap U) \geq n$ and $\dim_P(C \cap U) > d$.*

Proof. Define

$$\mathcal{V} = \{V \subset K : V \text{ is open and there exists an } F_\sigma \text{ set } F \subset V \text{ such that } \dim_T F \leq n-1 \text{ and } \dim_P(V \setminus F) \leq d\}.$$

The countable stability of topological dimension for F_σ sets and the countable stability of the packing dimension imply that \mathcal{V} is closed for taking countable unions. Let $V = \bigcup \mathcal{V}$, then the Lindelöf property of V implies that $V \in \mathcal{V}$. Clearly an open set U satisfies $U \in \mathcal{V}$ iff $U \subset V$. Let us define $C = K \setminus V$, then $d < d_P^n(K)$ yields that C is a non-empty compact set. Let U be an open set intersecting C , we need to show that $\dim_T(C \cap U) \geq n$ and $\dim_P(C \cap U) > d$. We prove the stronger statement that there is no F_σ set $F_1 \subset C \cap U$ such that

$$(4.2) \quad \dim_T F_1 \leq n-1 \quad \text{and} \quad \dim_P((C \cap U) \setminus F_1) \leq d.$$

Assume to the contrary that there exists such a set F_1 . As $U \setminus C \subset V$ is open, we have $U \setminus C \in \mathcal{V}$. Thus there exists an F_σ set $F_2 \subset U \setminus C$ such that

$$(4.3) \quad \dim_T F_2 \leq n-1 \quad \text{and} \quad \dim_P((U \setminus C) \setminus F_2) \leq d.$$

Then $F = F_1 \cup F_2$ is F_σ in U , and the countable stability of topological dimension for F_σ sets yields that $\dim_T F \leq n-1$. The countable stability of packing dimension and the second inequalities of (4.2) and (4.3) imply that

$$\dim_P(U \setminus F) = \max\{\dim_P((C \cap U) \setminus F_1), \dim_P((U \setminus C) \setminus F_2)\} \leq d.$$

Therefore F witnesses $U \in \mathcal{V}$, so $U \subset V$. Hence $C \cap U \subset C \cap V = \emptyset$, thus $C \cap U = \emptyset$, which is a contradiction. The proof is complete. \square

Proof of Theorem 4.11. Since the countable intersection of co-meager sets is co-meager, it is enough to prove the theorem for a fixed $d < d_P^n(K)$. By Lemma 4.15 and Corollary 2.13 we may assume that $\dim_T U \geq n$ and $\dim_P U > d$ for all non-empty open sets $U \subset K$. Applying Theorem 4.12 for $h(x) = x^d$ implies that for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \text{int } f(K)$ we have $\mathcal{P}^d(f^{-1}(y)) = \infty$, so $\dim_P f^{-1}(y) \geq d$. By Theorem 4.10 the set $U_f = \text{int } f(K)$ is not empty for a generic $f \in C(K, \mathbb{R}^n)$, which finishes the proof. \square

In the remaining part of this subsection we prove Theorem 4.12. First we need some preparation.

Definition 4.16. Assume that a compact metric space K and $n \in \mathbb{N}^+$ are given. For all $m \in \mathbb{N}^+$ define

$$\mathcal{D}_m = \{f \in C(K, \mathbb{R}^n) : \text{there exists an } \varepsilon > 0 \text{ such that} \\ g(K) \setminus B(\partial g(K), 1/m) \subset f(K) \text{ for all } g \in B(f, \varepsilon)\}.$$

If $f \in \mathcal{D}_m$ then there is a witness $\varepsilon(f, m) > 0$ corresponding to the definition.

The following lemma is [2, Lemma 7.11].

Lemma 4.17. *Let K be a compact metric space in which every non-empty open set is uncountable. Then $\mathcal{D}_m = \mathcal{D}_m(K, n)$ is dense in $C(K, \mathbb{R}^n)$ for all $m, n \in \mathbb{N}^+$.*

Lemma 4.18. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then for each $y \in \mathbb{R}^n$ and $r > 0$ there exists an onto map $f: K \rightarrow B(y, r)$ such that $B(y, r - t) \subset g(K)$ for all $0 < t < r$ and $g \in B(f, t)$.*

Proof. By [13, Theorem VI 2.] there exists a continuous map $f: K \rightarrow \mathbb{R}^n$ with a *stable value*, that is, there exist $y_0 \in \mathbb{R}^n$ and $r_0 > 0$ such that $y_0 \in g(K)$ for all $g \in B(f, r_0)$. We may assume that $y_0 = y$ and $r_0 = r$ by composing f with affine maps of \mathbb{R}^n . We show that for all $0 < t < r$ and $g \in B(f, t)$ we have $B(y, r - t) \subset g(K)$. Assume to the contrary that there is a $t \in (0, r)$, a map $g \in B(f, t)$, and $z \in B(y, r - t) \setminus g(K)$. Let $h = g + (y - z)$. Clearly $h \in B(g, r - t) \subset B(f, r)$, so $y \in h(K)$. Let $x \in K$ such that $h(x) = y$. Then $g(x) = h(x) - y + z = z$, so $z \in g(K)$, which is a contradiction.

In particular, $B(y, r - t) \subset f(K)$ for all $0 < t < r$. As $f(K)$ is closed, we have $B(y, r) \subset f(K)$. We may assume that $f(K) = B(y, r)$ by replacing f with $p \circ f$, where p is the radial projection of \mathbb{R}^n onto $B(y, r)$. This concludes the proof. \square

Lemma 4.19. *Let $n \in \mathbb{N}^+$ and let $\varepsilon > 0$. Assume that h is a left-continuous gauge function and K is a compact metric space such that $\dim_T U \geq n$ and $\mathcal{P}^h(U) = \infty$ for every non-empty open set $U \subset K$. Assume that $\mathcal{U} \subset C(K, \mathbb{R}^n)$ is open, and $V \subset \mathbb{R}^n$ is a non-empty open set such that $V \subset f(K)$ for some $f \in \mathcal{U}$. Then there is an open set $\mathcal{V} \subset \mathcal{U}$ and $N \in \mathbb{N}^+$, for each $1 \leq i \leq N$ there is an $N_i \in \mathbb{N}^+$ and an open set $V_i \subset \mathbb{R}^n$, and for each $1 \leq i \leq N$ and $1 \leq j \leq N_i$ there is a non-empty regular closed set $B_{ij} \subset K$ such that*

- (i) $V = \bigcup_{i=1}^N V_i$,
- (ii) $\text{diam } B_{ij} \leq \varepsilon$ for all $1 \leq i \leq N$ and $1 \leq j \leq N_i$,
- (iii) $B_{ij} \cap B_{k\ell} = \emptyset$ if $(i, j) \neq (k, \ell)$,
- (iv) $P_\varepsilon^h(S_i) \geq 1/\varepsilon$ whenever S_i meets B_{ij} for all $1 \leq j \leq N_i$,
- (v) $V_i \subset g(B_{ij})$ and $\text{diam } g(B_{ij}) \leq 6\varepsilon$ for all $g \in \mathcal{V}$, $1 \leq i \leq N$, and $1 \leq j \leq N_i$.

Proof. Fix $f \in \mathcal{U}$ such that $V \subset f(K)$. We may assume that $U(f, 5\varepsilon) \subset \mathcal{U}$, otherwise we replace ε by a smaller positive number. Since K is compact and f is uniformly continuous, there is a $\delta_1 > 0$ and there are finitely many distinct points $x_1, \dots, x_N \in K$ such that

$$(4.4) \quad K = \bigcup_{i=1}^N B(x_i, \delta_1)$$

and for all $i \in \{1, \dots, N\}$ we have

$$(4.5) \quad f(B(x_i, \delta_1)) \subset U(y_i, \varepsilon),$$

where $y_i = f(x_i)$. Let $V_i = U(y_i, \varepsilon) \cap V$, then clearly $\bigcup_{i=1}^N V_i \subset V$. By (4.5) we have $V \subset f(K) \cap V \subset \bigcup_{i=1}^N V_i$, so (i) holds.

Choose $0 < \delta_2 < \delta_1$ such that the balls $U(x_i, \delta_2)$ are disjoint. Fix an arbitrary $1 \leq i \leq N$. Since $\mathcal{P}^h(U(x_i, \delta_2)) = \infty$, there exists a finite set

$$Z_i = \{z_{ij} : 1 \leq j \leq N_i\} \subset U(x_i, \delta_2)$$

such that $P_\varepsilon^h(Z_i) > 1/\varepsilon$. By the left-continuity of h there exists a $0 < \delta_3 < \varepsilon/2$ such that $B(z_{ij}, \delta_3)$ are disjoint balls in $U(x_i, \delta_2)$ and $P_\varepsilon^h(S_i) \geq 1/\varepsilon$ for each set $S_i \subset K$ which meets $B(z_{ij}, \delta_3)$ for all $1 \leq j \leq N_i$. For every $j \in \{1, \dots, N_i\}$ define the non-empty regular closed set

$$B_{ij} = \text{cl } U(z_{ij}, \delta_3) \subset B(x_i, \delta_2).$$

Then $\text{diam } B_{ij} \leq 2\delta_3 < \varepsilon$ yields (ii). Since the sets $\{U(x_i, \delta_2) : 1 \leq i \leq N\}$ and for all $1 \leq i \leq N$ the sets $\{B(z_{ij}, \delta_3) : 1 \leq j \leq N_i\}$ consist of pairwise disjoint balls, we obtain (iii). The definition of δ_3 implies (iv).

As $\dim_T B_{ij} \geq n$, by Lemma 4.18 there are onto maps $f_{ij} : B_{ij} \rightarrow B(y_i, 2\varepsilon)$ such that

$$(4.6) \quad B(y_i, \varepsilon) \subset g_{ij}(B_{ij}) \quad \text{for all } g_{ij} \in B(f_{ij}, \varepsilon).$$

Tietze's extension theorem and (4.5) yield that for all $i \in \{1, \dots, N\}$ there exist continuous maps $F_i : B(x_i, \delta_1) \rightarrow B(y_i, 2\varepsilon)$ such that $F_i = f_{ij}$ on B_{ij} and $F_i = f$ on $B(x_i, \delta_1) \setminus U(x_i, \delta_2)$. Let $F(x) = F_i(x)$ for all $x \in B(x_i, \delta_1)$ and $i \in \{1, \dots, N\}$. The construction, (4.4), and (4.5) imply that $F \in C(K, \mathbb{R}^n)$ is well-defined and for all $1 \leq i \leq N$ we have

$$(4.7) \quad F(B(x_i, \delta_1)) \cup f(B(x_i, \delta_1)) = B(y_i, 2\varepsilon).$$

Clearly (4.4) and (4.7) yield that $F \in B(f, 4\varepsilon)$. Define $\mathcal{V} = U(F, \varepsilon)$, then we obtain $\mathcal{V} \subset U(f, 5\varepsilon) \subset \mathcal{U}$. Fix $1 \leq i \leq N$, $1 \leq j \leq N_i$, and $g \in \mathcal{V}$ arbitrarily. Then

$$g(B_{ij}) \subset B(f_{ij}(B_{ij}), \varepsilon) \subset B(y_i, 3\varepsilon),$$

so $\text{diam } g(B_{ij}) \leq 6\varepsilon$. Clearly $g|_{B_{ij}}$ satisfies $g_{ij} \in B(f_{ij}, \varepsilon)$, thus (4.6) yields

$$V_i \subset U(y_i, \varepsilon) \subset g_{ij}(B_{ij}) = g(B_{ij}),$$

so (v) holds. The proof is complete. \square

Now we are ready to prove Theorem 4.12.

Proof of Theorem 4.12. For each $m \in \mathbb{N}^+$ let

$$\mathcal{G}_m = \{f \in C(K, \mathbb{R}^n) : \mathcal{P}^h|_{f^{-1}(y)} \text{ is not } \sigma\text{-finite for all } y \in f(K) \setminus B(\partial f(K), 1/m)\}.$$

Fix $m \in \mathbb{N}^+$. It is enough to show that \mathcal{G}_m is co-meager, since then $\mathcal{G} = \bigcap_{m=1}^{\infty} \mathcal{G}_m$ will be our desired co-meager set in $C(K, \mathbb{R}^n)$. We will play the Banach-Mazur game in the complete metric space $C(K, \mathbb{R}^n)$: First Player I chooses a non-empty open set $\mathcal{U}_0 \subset C(K, \mathbb{R}^n)$, then Player II chooses a non-empty open set $\mathcal{V}_0 \subset \mathcal{U}_0$, Player I continues with a non-empty open set $\mathcal{U}_1 \subset \mathcal{V}_0$, Player II chooses a non-empty open set $\mathcal{V}_1 \subset \mathcal{U}_1$, and so on. By definition Player II wins this game if $\bigcap_{k=0}^{\infty} \mathcal{V}_k \subset \mathcal{G}_m$. It is well-known that Player II has a winning strategy iff \mathcal{G}_m is co-meager in $C(K, \mathbb{R}^n)$, see [21, Theorem 1] or [14, Theorem 8.33]. Thus we need to prove that Player II has a winning strategy.

Assume that the non-empty open set $\mathcal{U}_0 \subset C(K, \mathbb{R}^n)$ is given. By Lemma 4.17 we can fix $f_0 \in \mathcal{D}_m \cap \mathcal{U}_0$ and a witness $\varepsilon = \varepsilon(f_0, m) > 0$ corresponding to Definition 4.16. Let $\mathcal{V}_0 = B(f_0, \varepsilon) \cap \mathcal{U}_0$ and let $V_0 = \text{int } f_0(K)$. The definition of \mathcal{D}_m yields that for all $f \in \mathcal{V}_0$ we have

$$(4.8) \quad f(K) \setminus B(\partial f(K), 1/m) \subset V_0.$$

If $V_0 = \emptyset$ then clearly $\mathcal{V}_0 \subset \mathcal{G}_m$, and Player II wins independently of the subsequent moves. Therefore we may assume that $V_0 \subset \mathbb{R}^n$ is a non-empty open set.

Let $\mathcal{U}_1 \subset \mathcal{V}_0$ be given. By Lemma 4.19 there is a non-empty open set $\mathcal{V}_1 \subset \mathcal{U}_1$ and $N_0 \in \mathbb{N}^+$ such that for each $1 \leq i_1 \leq N_0$ there is an integer $N_{i_1} \in \mathbb{N}^+$ and an open set $V_{i_1} \subset \mathbb{R}^n$, and for each $1 \leq i_1 \leq N_0$ and $1 \leq i_2 \leq N_{i_1}$ there is a non-empty regular closed set $B_{i_1 i_2} \subset K$ such that

$$V_0 = \bigcup_{i_1=1}^{N_0} V_{i_1},$$

for all $1 \leq i_1 \leq N_0$ and $1 \leq i_2 \leq N_{i_1}$ we have

$$\text{diam } B_{i_1 i_2} \leq 1,$$

we have

$$B_{i_1 i_2} \cap B_{j_1 j_2} = \emptyset \quad \text{if } (i_1, i_2) \neq (j_1, j_2),$$

for all $1 \leq i_1 \leq N_0$ if S_{i_1} meets $B_{i_1 i}$ for all $1 \leq i \leq N_{i_1}$, then

$$P_1^h(S_{i_1}) \geq 1,$$

and for all $g \in \mathcal{V}_1$, $1 \leq i_1 \leq N_0$, and $1 \leq i_2 \leq N_{i_1}$ we have

$$V_{i_1} \subset g(B_{i_1 i_2}) \quad \text{and} \quad \text{diam } g(B_{i_1 i_2}) \leq 6.$$

We apply induction on k . If $B_{i_1 \dots i_{2k}} \subset B_{i_1 \dots i_{2k-2}} \subset \dots \subset B_{i_1 i_2} \subset K$ is a nested sequence such that $B_{i_1 \dots i_{2k}} \neq \emptyset$ and $B_{i_1 \dots i_{2\ell}}$ is a regular closed set in the relative topology of $B_{i_1 \dots i_{2\ell-2}}$ for each $1 \leq \ell \leq k$ (we adapt the convention $B_{i_1 \dots i_0} = K$), then Fact 2.1 implies that $\text{int } B_{i_1 \dots i_{2k}} \neq \emptyset$, thus $\dim_T B_{i_1 \dots i_{2k}} \geq n$. Therefore, assuming that the pairwise disjoint closed sets $B_{i_1 \dots i_{2k}} \subset K$ admitting the above nested sequence are given, we can apply Lemma 4.19 for them. This yields closed sets $B_{i_1 \dots i_{2k+2}} \subset B_{i_1 \dots i_{2k}}$ and non-empty open sets $\mathcal{V}_{i_1 \dots i_{2k+2}} \subset C(B_{i_1 \dots i_{2k+2}}, \mathbb{R}^n)$. Then by Tietze's extension theorem we can define the non-empty open set

$$\mathcal{V}_{k+1} = \mathcal{U}_{k+1} \cap \{f \in C(K, \mathbb{R}^n) : f|_{B_{i_1 \dots i_{2k+2}}} \in \mathcal{V}_{i_1 \dots i_{2k+2}} \text{ for all } i_1, \dots, i_{2k+2}\}.$$

That is, in the $(k+1)$ st step we can define a non-empty open set $\mathcal{V}_{k+1} \subset \mathcal{U}_{k+1}$, for each $1 \leq i_{2k+1} \leq N_{i_1 \dots i_{2k}}$ a non-empty open set $V_{i_1 \dots i_{2k+1}} \subset \mathbb{R}^n$ and a positive integer $N_{i_1 \dots i_{2k+1}}$, for every $1 \leq i_{2k+1} \leq N_{i_1 \dots i_{2k}}$ and $1 \leq i_{2k+2} \leq N_{i_1 \dots i_{2k+1}}$ a set $B_{i_1 \dots i_{2k+2}} \subset B_{i_1 \dots i_{2k}}$ such that $B_{i_1 \dots i_{2k+2}}$ is a non-empty regular closed set in the relative topology of $B_{i_1 \dots i_{2k}}$ and the following holds. For all $1 \leq i_{2k} \leq N_{i_1 \dots i_{2k-1}}$ we have

$$(4.9) \quad V_{i_1 \dots i_{2k-1}} = \bigcup_{i_{2k+1}=1}^{N_{i_1 \dots i_{2k}}} V_{i_1 \dots i_{2k+1}},$$

for all $1 \leq i_{2k+1} \leq N_{i_1 \dots i_{2k}}$ and $1 \leq i_{2k+2} \leq N_{i_1 \dots i_{2k+1}}$ we have

$$(4.10) \quad \text{diam } B_{i_1 \dots i_{2k+2}} \leq 2^{-k},$$

we have

$$(4.11) \quad B_{i_1 \dots i_{2k+2}} \cap B_{j_1 \dots j_{2k+2}} = \emptyset \quad \text{if } (i_1, \dots, i_{2k+2}) \neq (j_1, \dots, j_{2k+2}),$$

for all $1 \leq i_{2k+1} \leq N_{i_1 \dots i_{2k}}$ if $S_{i_1 \dots i_{2k+1}}$ meets $B_{i_1 \dots i_{2k+1}i}$ for all $1 \leq i \leq N_{i_1 \dots i_{2k+1}}$, then

$$(4.12) \quad \mathcal{P}_{2^{-k}}^h(S_{i_1 \dots i_{2k+1}}) \geq 2^k,$$

and for all $g \in \mathcal{V}_{k+1}$, $1 \leq i_{2k+1} \leq N_{i_1 \dots i_{2k}}$, and $1 \leq i_{2k+2} \leq N_{i_1 \dots i_{2k+1}}$ we have

$$(4.13) \quad V_{i_1 \dots i_{2k+1}} \subset g(B_{i_1 \dots i_{2k+2}}) \quad \text{and} \quad \text{diam } g(B_{i_1 \dots i_{2k+2}}) \leq 6 \cdot 2^{-k}.$$

The regularity of the closed sets $B_{i_1 \dots i_{2k}}$ and (4.11) are only needed to apply the induction, we will not use them later.

Fix $f \in \bigcap_{k=0}^{\infty} \mathcal{V}_k$, we need to show that $f \in \mathcal{F}$. Fix $y \in V_0$, by (4.8) it is enough to show that $\mathcal{P}^h|_{f^{-1}(y)}$ is not σ -finite. Define the set

$$C = \bigcap_{k=1}^{\infty} C_k, \quad \text{where} \quad C_k = \bigcup \{B_{i_1 \dots i_{2k}} : y \in V_{i_1 \dots i_{2k-1}}\}.$$

Then (4.13) implies that $f(x) = y$ for all $x \in C$, therefore $C \subset f^{-1}(y)$. Hence it is enough to prove that $\mathcal{P}^h|_C$ is not σ -finite. Fix an arbitrary $k \in \mathbb{N}^+$ and indices (i_1, \dots, i_{2k}) such that $y \in V_{i_1 \dots i_{2k-1}}$. By (4.10) every open set $U \subset K$ intersecting C contains a set $B_{i_1 \dots i_{2k}}$ such that $y \in V_{i_1 \dots i_{2k-1}}$, so by Lemma 2.9 it is enough to show that $P_0^h(C \cap B_{i_1 \dots i_{2k}}) = \infty$. Fix an arbitrary $\ell \geq k$, it is enough to prove that

$$(4.14) \quad P_{2^{-\ell}}^h(C \cap B_{i_1 \dots i_{2k}}) \geq 2^\ell.$$

By (4.9) we can fix indices $i_{2k+1}, \dots, i_{2\ell+1}$ such that $y \in V_{i_1 \dots i_{2\ell+1}}$. The definition of C and (4.9) imply that $C \cap B_{i_1 \dots i_{2\ell+1}i} \neq \emptyset$ for all $1 \leq i \leq N_{i_1 \dots i_{2\ell+1}}$. Let

$$S_{i_1 \dots i_{2\ell+1}} = \bigcup \{C \cap B_{i_1 \dots i_{2\ell+1}i} : 1 \leq i \leq N_{i_1 \dots i_{2\ell+1}}\}.$$

Then $S_{i_1 \dots i_{2\ell+1}} \subset C \cap B_{i_1 \dots i_{2k}}$, so (4.12) yields that

$$P_{2^{-\ell}}^h(C \cap B_{i_1 \dots i_{2k}}) \geq P_{2^{-\ell}}^h(S_{i_1 \dots i_{2\ell+1}}) \geq 2^\ell.$$

Thus (4.14) holds, and the proof is complete. \square

5. FIBERS ON FRACTALS

The main goal of this section is to prove Theorem 5.2. Recall the following notation.

Notation 5.1. Let $n \in \mathbb{N}^+$ and let K be a compact metric space. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . For each $f \in C(K, \mathbb{R}^n)$ let

$$R_*(f) = \{y \in f(K) : \dim_* f^{-1}(y) = d_*^n(K)\}.$$

Theorem 5.2. Let $n \in \mathbb{N}^+$ and assume that \dim_* is one of \dim_T , \dim_H , or \dim_P . Let K be a compact metric space with $d_*(K) > 0$. The following are equivalent:

- (i) $R_*(f) = \text{int } f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (ii) $R_*(f)$ is dense in $\text{int } f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (iii) $d_*^n(U) = d_*^n(K)$ or $\dim_T U < n$ for all open sets $U \subset K$.

We obtain the following corollary.

Corollary 5.3. Let $n \in \mathbb{N}^+$ and assume that \dim_* is one of \dim_T , \dim_H , or \dim_P . Let K be a compact metric space with $d_*^n(K) > 0$. The following are equivalent:

- (1) $R_*(f) = \text{int } f(K)$ and $\text{int } f(K)$ is dense in $f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (2) $R_*(f)$ is dense in $f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (3) $d_*^n(U) = d_*^n(K)$ for all non-empty open sets $U \subset K$.

Let $m \geq n$ be positive integers. Theorem 3.8 implies that $d_T^n([0, 1]^m) = m - n$, and Lemmas 3.10 and 3.11 yield that $d_H^n([0, 1]^m) = m - n$ and $d_P^n([0, 1]^m) = m$. Thus Theorem 5.2 implies the following extension of Theorem 1.1.

Corollary 5.4. Let $m, n \in \mathbb{N}^+$ with $m \geq n$. For a generic $f \in C([0, 1]^m, \mathbb{R}^n)$ for all $y \in \text{int } f([0, 1]^m)$ we have

$$\dim_T f^{-1}(y) = \dim_H f^{-1}(y) = m - n \quad \text{and} \quad \dim_P f^{-1}(y) = m.$$

Before proving Theorem 5.2 we need some preparation. The proof of (iii) \implies (i) will basically follow the proof of [2, Theorem 7.9], the new ingredient will be the application of Lemma 5.10. First we need some lemmas.

Lemma 5.5. Let X, Z be Polish spaces, and let $E \subset X \times Z$ be a Borel set. Assume that E_x is compact for all $x \in X$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . Define

$$d_* : X \rightarrow [-1, \infty], \quad d_*(x) = \dim_* E_x.$$

Then d_* is measurable for $\sigma(\mathbf{A})$.

Proof. By [19, Theorem 6.1] the map d_H is Borel measurable, and [19, Theorem 6.4] yields that d_P is measurable for $\sigma(\mathbf{A})$. Thus it is enough to prove that d_T is Borel measurable. Define

$$D_T : \mathcal{K}(K) \rightarrow [-1, \infty], \quad D_T(C) = \dim_T C,$$

where $\mathcal{K}(K)$ is the set of compact subsets of K endowed with the Hausdorff metric, see [14, Section 4.F] for the definition. By [17, page 108, Theorem 4] the map D_T is Borel measurable. Define $S : X \rightarrow \mathcal{K}(K)$ as $S(x) = E_x$. Then S is Borel measurable by [14, Theorem 28.8]. Therefore the map $d_T = D_T \circ S$ is Borel measurable, too. \square

Remark 5.6. Unlike [19], we adopt the convention $\dim_* \emptyset = -1$, which modifies some fibers of d_* by $\{x \in X : E_x = \emptyset\} = (\text{pr}_X(E))^c$, where pr_X denotes the natural projection of $X \times Z$ onto X . In order to fix this problem we show that $\text{pr}_X(E)$ is Borel. As E is Borel and E_x are compact, this follows from [14, Theorem 18.18].

For the following assume that a compact metric space K and the numbers $n \in \mathbb{N}^+$ and $d \in \mathbb{R}$ are given. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . The proof of the following lemma is analogous to that of [4, Lemma 2.11].

Lemma 5.7. *The set*

$$\Delta = \{(f, y) \in C(K, \mathbb{R}^n) \times \mathbb{R}^n : \dim_* f^{-1}(y) < d\}$$

is in $\sigma(\mathbf{A})$.

Proof. Let $X = C(K, \mathbb{R}^n) \times \mathbb{R}^n$, let $Z = K$, and define

$$E = \{(f, y, z) \in C(K, \mathbb{R}^n) \times \mathbb{R}^n \times K : f(z) = y\} \subseteq X \times Z.$$

Clearly X, Z are Polish spaces and E is closed, thus Borel. For each $(f, y) \in X$ the set $E_{(f, y)} = \{z \in K : f(z) = y\} = f^{-1}(y)$ is compact. Thus Lemma 5.5 yields that the map

$$d_* : X \rightarrow [0, \infty], \quad d_*((f, y)) = \dim_* E_{(f, y)} = \dim_* f^{-1}(y)$$

is measurable for $\sigma(\mathbf{A})$. Therefore

$$d_*^{-1}((-\infty, d)) = \{(f, y) \in C(K, \mathbb{R}^n) \times \mathbb{R}^n : \dim_* f^{-1}(y) < d\} = \Delta$$

is in $\sigma(\mathbf{A})$. □

Definition 5.8. For all $0 < r_1 < r_2$ and $y_0 \in \mathbb{R}^n$ define

$$\begin{aligned} \mathcal{H}(r_1, r_2, y_0) &= \{f \in C(K, \mathbb{R}^n) : f(K) \subset B(y_0, r_2) \text{ and} \\ &\quad \dim_* f^{-1}(y) \geq d \text{ for all } y \in B(y_0, r_1)\}. \end{aligned}$$

Lemma 5.9. *The sets $\mathcal{H}(r_1, r_2, y_0)$ have the Baire property.*

Proof. Let $\mathcal{H} = \mathcal{H}(r_1, r_2, y_0)$ and let $\text{pr} : C(K, \mathbb{R}^n) \times \mathbb{R}^n \rightarrow C(K, \mathbb{R}^n)$ be the natural projection of $C(K, \mathbb{R}^n) \times \mathbb{R}^n$ onto $C(K, \mathbb{R}^n)$. Define

$$\Delta = \{(f, y) \in C(K, \mathbb{R}^n) \times \mathbb{R}^n : \dim_* f^{-1}(y) < d\},$$

by Lemma 5.7 the set Δ is in $\sigma(\mathbf{A})$. It is easy to see that $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2$, where

$$\begin{aligned} \mathcal{H}_1 &= \{f \in C(K, \mathbb{R}^n) : f(K) \subset B(y_0, r_2)\}, \\ \mathcal{H}_2 &= (\text{pr}(\Delta \cap (C(K, \mathbb{R}^n) \times B(y_0, r_1))))^c. \end{aligned}$$

Clearly \mathcal{H}_1 is closed. As $\sigma(\mathbf{A})$ is closed under taking intersection, projection, and complement, we obtain that \mathcal{H}_2 is in $\sigma(\mathbf{A})$. Therefore \mathcal{H} is in $\sigma(\mathbf{A})$, so it has the Baire property. □

Lemma 5.10. *If $\mathcal{H}(r_1, r_2, y_0)$ is of second category for some parameters r_1, r_2, y_0 , then it is of second category for all parameters.*

Proof. Assume that $0 < r_1 < r_2$, $0 < q_1 < q_2$, and $y_0, z_0 \in \mathbb{R}^n$. We need to prove that if $\mathcal{H}(r_1, r_2, y_0)$ is of second category, then $\mathcal{H}(q_1, q_2, z_0)$ is of second category, too. Clearly it is enough to find a homeomorphism $H : C(K, \mathbb{R}^n) \rightarrow C(K, \mathbb{R}^n)$ such that $H(\mathcal{H}(r_1, r_2, y_0)) = \mathcal{H}(q_1, q_2, z_0)$. Define a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $h(B(y_0, r_i)) = B(z_0, q_i)$ for $i \in \{1, 2\}$, and h is affine on $\mathbb{R}^n \setminus B(y_0, r_2)$. Then h and h^{-1} are uniformly continuous, so

$$H : C(K, \mathbb{R}^n) \rightarrow C(K, \mathbb{R}^n), \quad H(f) = h \circ f$$

is a homeomorphism with $H(\mathcal{H}(r_1, r_2, y_0)) = \mathcal{H}(q_1, q_2, z_0)$. □

Now we are ready to prove Theorem 5.2.

Proof of Theorem 5.2. The implication (i) \implies (ii) is straightforward.

Now we prove (ii) \implies (iii). Assume to the contrary that there exists an open set $U \subset K$ such that $\dim_T U \geq n$ and $d_*^n(U) < d_*^n(K)$. As U is a countable union of compact sets, the countable stability of topological dimension for closed sets yields that there is a compact set $C \subset U$ such that $\dim_T C \geq n$. Let D be a compact set such that $C \subset \text{int } D \subset D \subset U$. Then clearly $d_*^n(D) < d_*^n(K)$. Define the sets

$$\begin{aligned}\mathcal{A} &= \{f \in C(K, \mathbb{R}^n) : \text{int } f(C) \neq \emptyset\}, \\ \mathcal{B} &= \{f \in C(K, \mathbb{R}^n) : \dim_*(f^{-1}(y) \cap D) \leq d_*^n(D) \text{ for all } y \in \mathbb{R}^n\}, \\ \mathcal{C} &= \{f \in C(K, \mathbb{R}^n) : f(C) \cap f(K \setminus \text{int } D) = \emptyset\}.\end{aligned}$$

As $\dim_T C \geq n$, Theorem 4.10 and Corollary 2.13 imply that \mathcal{A} is co-meager. The Main Theorem and Corollary 2.13 yield that \mathcal{B} is co-meager, too. Tietze's extension theorem implies that there are distinct points $y_1, y_2 \in \mathbb{R}^n$ and there is a continuous map $f \in C(K, \mathbb{R}^n)$ such that $f(C) = y_1$ and $f(K \setminus \text{int } D) = y_2$. Let $r = |y_1 - y_2|/2 > 0$, then clearly $U(f, r) \subset \mathcal{C}$, so \mathcal{C} is of second category. Define

$$\mathcal{F} = \mathcal{A} \cap \mathcal{B} \cap \mathcal{C},$$

then \mathcal{F} is of second category. Let $f \in \mathcal{F}$ and let $U_f = \text{int } f(C) \subset \text{int } f(K)$. Since $f \in \mathcal{A}$, the set U_f is non-empty open in $\text{int } f(K)$. As $f \in \mathcal{C}$, we have $f^{-1}(y) \subset D$ for all $y \in U_f$. Therefore $f \in \mathcal{B}$ implies that for all $y \in U_f$ we have

$$\dim_* f^{-1}(y) = \dim_*(f^{-1}(y) \cap D) \leq d_*^n(D) < d_*^n(K).$$

Thus $R_*(f) \cap U_f = \emptyset$, so $R_*(f) \cap \text{int } K$ is not dense in $\text{int } K$. This contradicts (ii), so the implication (ii) \implies (iii) follows.

Finally, we show that (iii) \implies (i). Theorem 4.14 implies that $R_* \subset \text{int } f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$, and the Main Theorem yields that $\dim_* f^{-1}(y) \leq d_*^n(K)$ for all $y \in \mathbb{R}^n$. Therefore it is enough to prove that $\dim_* f^{-1}(y) \geq d_*^n(K)$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \text{int } f(K)$. By Lemma 2.15 and Corollary 2.13 we may assume that $\dim_T U \geq n$ for all non-empty open sets $U \subset K$. Choose a sequence $d_m \nearrow d_*^n(K)$. Let us fix $m \in \mathbb{N}^+$ and define

$$\mathcal{G}_m = \{f \in C(K, \mathbb{R}^n) : \dim_* f^{-1}(y) \geq d_m \text{ for all } y \in f(K) \setminus B(\partial f(K), 1/m)\}.$$

It is sufficient to verify that \mathcal{G}_m is co-meager, since then $\mathcal{G} = \bigcap_{m=1}^{\infty} \mathcal{G}_m$ will be our desired co-meager set in $C(K, \mathbb{R}^n)$. It is enough to prove that in every non-empty open subset of $C(K, \mathbb{R}^n)$ the set \mathcal{G}_m contains a set which is of second category and has the Baire property. As $\dim_T U \geq n$ for every non-empty open set $U \subset K$, we can apply Lemma 4.17. Let us fix an arbitrary $f_0 \in \mathcal{D}_m$ and a witness $\varepsilon = \varepsilon(f_0, m) > 0$ corresponding to Definition 4.16. As \mathcal{D}_m is dense in $C(K, \mathbb{R}^n)$ by Lemma 4.17, it is enough to show that $\mathcal{G}_m \cap B(f_0, \varepsilon)$ contains a set which is of second category and has the Baire property.

Since K is compact and f_0 is uniformly continuous, there is a $\delta_1 > 0$ and there are finitely many distinct points $x_1, \dots, x_k \in K$ such that

$$(5.1) \quad K = \bigcup_{i=1}^k B(x_i, \delta_1)$$

and for each $i \in \{1, \dots, k\}$ the oscillation of f_0 on $B(x_i, \delta_1)$ is less than $\varepsilon/4$. Choose $0 < \delta_2 < \delta_1$ such that the balls $K_i = B(x_i, \delta_2)$ are disjoint. Then clearly

$$(5.2) \quad f_0(K) \subset \bigcup_{i=1}^k B(f_0(x_i), \varepsilon/4).$$

Fix $i \in \{1, \dots, k\}$. According to Definition 5.8 for all $0 < r_1 < r_2$ and $y_0 \in \mathbb{R}^n$ let

$$\mathcal{H}_i(r_1, r_2, y_0) = \{f \in C(K_i, \mathbb{R}^n) : f(K_i) \subset B(y_0, r_2) \text{ and} \\ \dim_* f^{-1}(y) \geq d_m \text{ for all } y \in B(y_0, r_1)\},$$

and define

$$\mathcal{H}_i = \mathcal{H}_i(\varepsilon/4, \varepsilon/2, f_0(x_i)).$$

Lemma 5.9 yields that \mathcal{H}_i has the Baire property. Now we prove that \mathcal{H}_i is of second category. As $\dim_T K_i \geq n$ and $d_*^n(K_i) = d_*^n(K)$, the Main Theorem implies that for a generic $f \in C(K_i, \mathbb{R}^n)$ there exists a non-empty open set $U_{f, d_m} \subset \mathbb{R}^n$ such that $\dim_* f^{-1}(y) \geq d_m$ for all $y \in U_{f, d_m}$. Thus Baire's category theorem yields that there exist $0 < r_1 < r_2$ and $y_0 \in \mathbb{R}^n$ such that $\mathcal{H}_i(r_1, r_2, y_0)$ is of second category. Then Lemma 5.10 implies that \mathcal{H}_i is of second category.

Clearly $\mathcal{H}_i \subset B(f_0|_{K_i}, \varepsilon)$. Set

$$\mathcal{F} = \bigcap_{i=1}^k \mathcal{F}_i, \quad \text{where} \quad \mathcal{F}_i = \{f \in B(f_0, \varepsilon) : f|_{K_i} \in \mathcal{H}_i\}.$$

As the sets \mathcal{H}_i have the Baire property, \mathcal{F} has the Baire property, too. Clearly $\mathcal{F} \subset B(f_0, \varepsilon)$, and repeating the proof of [4, Lemma 3.8] verbatim yields that \mathcal{F} is of second category. Therefore, it is enough to show that $\mathcal{F} \subset \mathcal{G}_m$. Assume that $f \in \mathcal{F}$ and $y \in f(K) \setminus B(\partial f(K), 1/m)$, we need to prove that $\dim_* f^{-1}(y) \geq d_m$. The definition of $\varepsilon = \varepsilon(f_0, m)$ and $f \in B(f_0, \varepsilon)$ yield that $y \in f_0(K)$. By (5.2) there exists $i \in \{1, \dots, k\}$ such that $y \in B(f_0(x_i), \varepsilon/4)$. Then $f|_{K_i} \in \mathcal{H}_i$ implies that $\dim_* f^{-1}(y) \geq \dim_*(f|_{K_i})^{-1}(y) \geq d_m$. This completes the proof. \square

Remark 5.11. In the case of packing dimension the implication (iii) \implies (i) simply follows from Theorem 4.12. However, Lemma 5.7 is not superfluous even in the case of packing dimension, we will use it to prove Theorem 7.2.

Proof of Corollary 5.3. The implication (1) \implies (2) is straightforward.

Now we prove (2) \implies (3). Assume to the contrary that there is a non-empty open set $U \subset K$ such that $d_*^n(U) < d_*^n(K)$. Theorem 5.2 implies that $\dim_T U < n$. Let C, D be non-empty compact sets such that $C \subset \text{int } D \subset D \subset U$. Define

$$\mathcal{A} = \{f \in C(K, \mathbb{R}^n) : \dim_*(f^{-1}(y) \cap D) \leq 0 \text{ for all } y \in \mathbb{R}^n\}, \\ \mathcal{B} = \{f \in C(K, \mathbb{R}^n) : f(C) \cap f(K \setminus \text{int } D) = \emptyset\}.$$

Theorem 4.1 and Corollary 2.13 imply that \mathcal{A} is prevalent. Repeating the arguments of the proof Theorem 5.2 yields that \mathcal{B} is of second category, so $\mathcal{A} \cap \mathcal{B}$ is of second category, too. Fix $f \in \mathcal{A} \cap \mathcal{B}$ and let $U_f = f(K) \setminus f(K \setminus \text{int } D)$. Clearly U_f is relatively open in $f(K)$, and $f \in \mathcal{B}$ implies that $U_f \neq \emptyset$. Let $y \in U_f$ be arbitrarily fixed. Clearly $f^{-1}(y) \subset D$, so $f \in \mathcal{A}$ yields that

$$\dim_* f^{-1}(y) = \dim_*(f^{-1}(y) \cap D) = 0 < d_*^n(K).$$

Thus $R_*(f) \cap U_f = \emptyset$, so $R_*(f)$ is not dense in $f(K)$, which contradicts (2).

Finally, we show (3) \implies (1). By Theorem 5.2 it is enough to prove that $\text{int } f(K)$ is dense in $f(K)$ for a generic $f \in C(K, \mathbb{R}^n)$. Let \mathcal{D} be a countable dense subset of K and let $\mathcal{B} = \{B(x, 1/i) : x \in \mathcal{D}, i \in \mathbb{N}^+\}$. Let $B \in \mathcal{B}$ be given. As $d_*^n(B) = d_*^n(K)$, Fact 3.4 yields that $\dim_T B \geq n$. Therefore Theorem 4.10 and Corollary 2.13 imply that $\text{int } f(B) \neq \emptyset$ for a generic $f \in C(K, \mathbb{R}^n)$. As a countable intersection of co-meager sets is co-meager, for a generic $f \in C(K, \mathbb{R}^n)$ for all $B \in \mathcal{B}$ we have $\text{int } f(B) \neq \emptyset$, so $\text{int } f(K)$ is dense in $f(K)$. The proof is complete. \square

6. DIMENSIONS OF THE BOUNDARY OF GENERIC IMAGES

The goal of this section is to prove the following theorem.

Theorem 6.1. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\dim_T \partial f(K) = \dim_H \partial f(K) = n - 1.$$

Moreover, let h be a gauge function with $\lim_{r \rightarrow 0^+} h(r)/r^{n-1} = 0$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have $\mathcal{H}^h(\partial f(K)) = 0$ and $\mathcal{H}^{n-1}(\partial f(K)) > 0$.

Recall the following notation.

Notation 6.2. Let $n \in \mathbb{N}^+$ and let K be a compact metric space. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . For each $f \in C(K, \mathbb{R}^n)$ let

$$S_*(f) = \{y \in f(K) : \dim_* f^{-1}(y) < d_*^n(K)\}.$$

Theorems 5.2 and 6.1 imply the following.

Corollary 6.3. *Let $n \in \mathbb{N}^+$ and assume that \dim_* is one of \dim_T , \dim_H , or \dim_P . Let K be a compact metric space with $d_*^n(K) > 0$. Exactly one of the following holds:*

- (a) $\dim_H S_*(f) = n - 1$ for a generic $f \in C(K, \mathbb{R}^n)$;
- (b) $\text{int } S_*(f) \neq \emptyset$ for a generic $f \in C(K, \mathbb{R}^n)$.

Moreover, (a) is equivalent to the statements of Theorem 5.2.

First we need some preparation. The next theorem is [6, Theorem 2.1], which generalizes Theorem 2.14.

Theorem 6.4 (Balka-Farkas-Fraser-Hyde). *Let $n \in \mathbb{N}^+$ and let K be a compact metric space. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\dim_T f(K) = \dim_H f(K) = \min\{\dim_T K, n\}.$$

We need the following better upper bound, the proof is quite standard.

Theorem 6.5. *Let $m, n \in \mathbb{N}^+$ with $m < n$. Let K be a compact metric space and let h be a gauge function with $\dim_T K = m$ and $\lim_{r \rightarrow 0^+} h(r)/r^m = 0$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\mathcal{H}^h(f(K)) = 0.$$

Proof. Let $g: [0, \infty) \rightarrow [0, \infty)$ be the right-continuous modification of h defined as $g(r) = \lim_{t \rightarrow r^+} h(t)$. Clearly g is non-decreasing and $\lim_{r \rightarrow 0^+} h(r)/r^m = 0$ implies that $g(0) = 0$ and $\lim_{r \rightarrow 0^+} g(r)/r^m = 0$. As $g(r) \geq h(r)$ for all $r \geq 0$, it is enough to prove that $\mathcal{H}^g(f(K)) = 0$ for a generic $f \in C(K, \mathbb{R}^n)$. Let

$$\mathcal{F} = \{f \in C(K, \mathbb{R}^n) : \mathcal{H}^g(f(K)) = 0\},$$

and for all $i \in \mathbb{N}^+$ define

$$\mathcal{F}_i = \left\{ f \in C(K, \mathbb{R}^n) : \begin{array}{l} \text{there are open sets } U_1, \dots, U_k \subset \mathbb{R}^n \\ \text{such that } f(K) \subset \bigcup_{i=1}^k U_i \text{ and } \sum_{i=1}^k g(\text{diam } U_i) < 1/i \end{array} \right\}.$$

The sets \mathcal{F}_i are clearly open. As K is compact and g is right-continuous, we obtain $\mathcal{F} = \bigcap_{i=1}^{\infty} \mathcal{F}_i$. Thus \mathcal{F} is G_δ , so it is enough to prove that \mathcal{F} is dense in $C(K, \mathbb{R}^n)$. The set

$$\mathcal{G} = \{ f \in C(K, \mathbb{R}^n) : f(K) \text{ is contained in an } m\text{-dimensional polyhedron} \}$$

is dense in $C(K, \mathbb{R}^n)$, see [8, Chapter 1.10]. Clearly $\mathcal{G} \subset \mathcal{F}$, hence \mathcal{F} is dense in $C(K, \mathbb{R}^n)$, too. The proof is complete. \square

The hard part of Theorem 6.1 is the following.

Theorem 6.6. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space such that we have $\dim_T U \geq n$ for all non-empty open sets $U \subset K$. Let h be a gauge function with $\lim_{r \rightarrow 0^+} h(r)/r^{n-1} = 0$. Then for a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\mathcal{H}^h(\partial f(K)) = 0.$$

First we deduce Theorem 6.1 from Theorems 6.5 and 6.6.

Proof of Theorem 6.1. We prove the lower bounds first. Theorem 4.10 yields that for a generic $f \in C(K, \mathbb{R}^n)$ we have $\text{int } f(K) \neq \emptyset$. First fix such $f \in C(K, \mathbb{R}^n)$. By [7, Theorem 1.8.12] the boundary of a bounded, non-empty open set in \mathbb{R}^n has topological dimension at least $n-1$, so $\dim_T \partial(\text{int } f(K)) \geq n-1$. Clearly $\partial(\text{int } f(K)) \subset \partial f(K)$, so $\dim_T \partial f(K) \geq n-1$. Let $\text{pr}: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ denote the orthogonal projection of \mathbb{R}^n onto its first $n-1$ coordinates (where $\mathbb{R}^0 = \{0\}$ by convention), then $\text{pr}(f(K))$ contains a non-empty open set, and clearly we have $\text{pr}(\partial f(K)) = \text{pr}(f(K))$. As Hausdorff measures cannot increase under projections, we obtain

$$\mathcal{H}^{n-1}(\partial f(K)) \geq \mathcal{H}^{n-1}(\text{pr}(\partial f(K))) = \mathcal{H}^{n-1}(\text{pr}(f(K))) > 0.$$

Now we prove the upper bounds. It is enough to show that $\mathcal{H}^h(\partial f(K)) = 0$ for a generic $f \in C(K, \mathbb{R}^n)$, then the choice $h(x) = x^{n-1}/\log(1/x)$ and Theorem 3.6 will immediately imply that

$$\dim_T \partial f(K) \leq \dim_H \partial f(K) \leq n-1.$$

Let $U \subset K$ be the maximal open set such that $\dim_T U < n$ and let $C = K \setminus U$. By Lemma 2.15 we have $\dim_T(C \cap V) \geq n$ for all open sets $V \subset K$ intersecting C . Assume that $U = \bigcup_{i=1}^{\infty} K_i$, where $K_i \subset K$ are compact. As $\dim_T K_i \leq n-1$ and a countable intersection of co-meager sets is co-meager, Theorem 6.5, Theorem 6.6, and Corollary 2.13 imply that for a generic $f \in C(K, \mathbb{R}^n)$ we have $\mathcal{H}^h(f(K_i)) = 0$ for all $i \geq 1$ and $\mathcal{H}^h(\partial f(C)) = 0$. Clearly

$$\partial f(K) \subset \bigcup_{i=1}^{\infty} f(K_i) \cup \partial f(C),$$

so the countably subadditivity of \mathcal{H}^h implies that $\mathcal{H}^h(\partial f(K)) = 0$ for a generic $f \in C(K, \mathbb{R}^n)$. The proof is complete. \square

Before proving Theorem 6.6 we need some more preparation.

Definition 6.7. For a compact metric space K and $n \in \mathbb{N}^+$ define

$$\mathcal{G}_n(K) = \left\{ g \in C(K, \mathbb{R}^n) : g(K) = \bigcup_{i=1}^k B(y_i, r) \text{ for some } k, y_i \in \mathbb{R}^n, \text{ and } r > 0 \right. \\ \left. \text{such that } \bigcup_{i=1}^k B(y_i, r-t) \subset f(K) \text{ for all } 0 < t < r \text{ and } f \in B(g, t) \right\}.$$

The proof of the following lemma is similar to that of Lemma 4.19.

Lemma 6.8. Let $n \in \mathbb{N}^+$ and let K be a compact metric space such that we have $\dim_T U \geq n$ for all non-empty open sets $U \subset K$. Then $\mathcal{G}_n(K)$ is dense in $C(K, \mathbb{R}^n)$.

Proof. Assume that $f_0 \in C(K, \mathbb{R}^n)$ and $r > 0$ are given, we need to show that $\mathcal{G}_n(K) \cap B(f_0, 2r) \neq \emptyset$. Since K is compact and f_0 is uniformly continuous, there is a $\delta > 0$ and there are finitely many distinct points $x_1, \dots, x_k \in K$ such that

$$(6.1) \quad K = \bigcup_{i=1}^k B(x_i, \delta)$$

and for all $i \in \{1, \dots, k\}$ we have

$$(6.2) \quad f_0(B(x_i, \delta)) \subset B(y_i, r),$$

where $y_i = f_0(x_i)$. Choose $0 < 2\varepsilon < \delta$ such that the balls $B(x_i, 2\varepsilon)$ are disjoint and let $K_i = B(x_i, \varepsilon)$ for all $i \in \{1, \dots, k\}$. As $\dim_T K_i \geq n$, by Lemma 4.18 there exist onto maps $g_i: K_i \rightarrow B(y_i, r)$ such that $B(y_i, r-t) \subset f_i(K_i)$ for all $0 < t < r$ and $f_i \in B(g_i, t)$. Then (6.2) and Tietze's extension theorem yield that there are continuous maps $G_i: B(x_i, \delta) \rightarrow B(y_i, r)$ such that $G_i = g_i$ on K_i and $G_i = f_0$ on $B(x_i, \delta) \setminus U(x_i, 2\varepsilon)$. Let $g(x) = G_i(x)$ for all $x \in B(x_i, \delta)$ and $i \in \{1, \dots, k\}$. Then the construction and (6.1) imply that $g \in C(K, \mathbb{R}^n)$ is well-defined and $g(B(x_i, \delta)) = B(y_i, r)$ for all i . Therefore

$$g(K) = \bigcup_{i=1}^k B(y_i, r).$$

The construction and (6.2) imply that

$$f_0(B(x_i, \delta)) \cup g(B(x_i, \delta)) = B(y_i, r),$$

which yields that $g \in B(f_0, 2r)$.

Finally, assume that $0 < t < r$ and $f \in B(g, t)$. Let $f_i = f|_{K_i}$, then clearly $f_i \in B(g_i, t)$, so for all $i \in \{1, \dots, k\}$ we have

$$B(y_i, r-t) \subset f_i(K_i) \subset f(K).$$

Hence $g \in \mathcal{G}_n(K)$, and the proof is complete. \square

Lemma 6.9. For each $n \in \mathbb{N}^+$ there is a finite constant c_n such that the set $B(y, r + \varepsilon) \setminus U(y, r - \varepsilon)$ can be covered by $c_n((r + 2\varepsilon)/\varepsilon)^{n-1}$ sets of diameter ε for all $y \in \mathbb{R}^n$ and $0 < 2\varepsilon < r$.

Proof. Let $n \in \mathbb{N}^+$, $y \in \mathbb{R}^n$, and $0 < 2\varepsilon < r$ be arbitrarily fixed. Let

$$R_1 = B(y, r + \varepsilon) \setminus U(y, r - \varepsilon) \quad \text{and} \quad R_2 = B(y, r + 2\varepsilon) \setminus U(y, r - 2\varepsilon).$$

Define

$$\delta = \frac{\varepsilon}{\sqrt{n}} \quad \text{and} \quad \mathcal{Q}_\delta = \left\{ \prod_{i=1}^n [k_i \delta, (k_i + 1) \delta] : k_i \in \mathbb{Z} \right\}.$$

Clearly

$$(r + 2\varepsilon)^n - (r - 2\varepsilon)^n = 4\varepsilon \sum_{k=0}^{n-1} (r + 2\varepsilon)^{n-1-k} (r - 2\varepsilon)^k \leq 4\varepsilon n (r + 2\varepsilon)^{n-1}.$$

For each $Q \in \mathcal{Q}_\delta$ we have $\text{diam } Q = \varepsilon$, and if $Q \cap R_1 \neq \emptyset$ then $Q \subset R_2$. Thus the above inequality yields that the number of sets of diameter ε needed to cover R_1 is at most

$$\begin{aligned} \#\{Q \in \mathcal{Q}_\delta : Q \cap R_1 \neq \emptyset\} &\leq \#\{Q \in \mathcal{Q}_\delta : Q \subset R_2\} \\ &\leq \delta^{-n} \mathcal{L}^n(R_2) \\ &= \varepsilon^{-n} n^{n/2} e_n ((r + 2\varepsilon)^n - (r - 2\varepsilon)^n) \\ &\leq 4e_n n^{n/2+1} \left(\frac{r + 2\varepsilon}{\varepsilon} \right)^{n-1} \\ &= c_n \left(\frac{r + 2\varepsilon}{\varepsilon} \right)^{n-1}, \end{aligned}$$

where \mathcal{L}^n denotes the n -dimensional Lebesgue measure, $e_n = \mathcal{L}^n(B(\mathbf{0}, 1))$, and $c_n = 4e_n n^{n/2+1}$. The proof is complete. \square

Now we are ready to prove Theorem 6.6.

Proof of Theorem 6.6. For all $g \in \mathcal{G}_n(K)$ choose $k(g) \in \mathbb{N}^+$ and $r(g) > 0$ according to Definition 6.7, and for each $m \in \mathbb{N}^+$ let $\varepsilon(g, m)$ be a small enough positive number to be chosen later such that $\varepsilon(g, m) < r(g)/2$ and $\lim_{m \rightarrow \infty} \varepsilon(g, m) = 0$. Define

$$\mathcal{F} = \bigcap_{m=1}^{\infty} \mathcal{F}_m, \quad \text{where} \quad \mathcal{F}_m = \bigcup_{g \in \mathcal{G}_n(K)} U(g, \varepsilon(g, m)).$$

Lemma 6.8 yields that \mathcal{F}_m is a dense open set in $C(K, \mathbb{R}^n)$ for all m , so \mathcal{F} is co-meager in $C(K, \mathbb{R}^n)$. We will prove that $\mathcal{H}^h(\partial f(K)) = 0$ for each $f \in \mathcal{F}$. Fix $f \in \mathcal{F}$ and $m \in \mathbb{N}^+$, it is enough to prove that $\varepsilon = \varepsilon(g, m)$ satisfies

$$(6.3) \quad \mathcal{H}_\varepsilon^h(\partial f(K)) \leq \frac{1}{m}.$$

Fix $g \in \mathcal{G}_n(K)$ such that $f \in U(g, \varepsilon(g, m))$. Let $\varepsilon = \varepsilon(g, m)$, $r = r(g)$, and $k = k(g)$. By the definition of $\mathcal{G}_n(K)$ we have

$$\bigcup_{i=1}^k B(y_i, r - \varepsilon) \subset f(K) \subset \bigcup_{i=1}^k B(y_i, r + \varepsilon),$$

thus

$$(6.4) \quad \partial f(K) \subset \bigcup_{i=1}^k (B(y_i, r + \varepsilon) \setminus U(y_i, r - \varepsilon)).$$

By Lemma 6.9 there is a constant $c_n \in \mathbb{N}^+$ such the sets $B(y_i, r + \varepsilon) \setminus U(y_i, r - \varepsilon)$ can be covered by $c_n((r + 2\varepsilon)/\varepsilon)^{n-1}$ sets of diameter ε . Now define ε such that $0 < \varepsilon < \min\{r/2, 1/m\}$ and

$$(6.5) \quad \frac{h(\varepsilon)}{\varepsilon^{n-1}} \leq \frac{(r + 2\varepsilon)^{1-n}}{kc_n m}.$$

Therefore (6.4) and (6.5) imply that

$$\mathcal{H}_\varepsilon^h(\partial f(K)) \leq kc_n \left(\frac{r + 2\varepsilon}{\varepsilon} \right)^{n-1} h(\varepsilon) \leq \frac{1}{m},$$

so (6.3) holds. The proof is complete. \square

7. FIBERS OF MAXIMAL DIMENSION

The Main Theorem implies the following.

Corollary 7.1. *Assume that K is a compact metric space and $n \in \mathbb{N}^+$ such that $n \leq \dim_T K < \infty$. Then for a generic $f \in C(K, \mathbb{R}^n)$ there is a non-empty open set $U_f \subset \mathbb{R}^n$ such that for all $y \in U_f$ we have*

$$\dim_T f^{-1}(y) = d_T^n(K).$$

In particular, for a generic $f \in C(K, \mathbb{R}^n)$ we have

$$\max\{\dim_T f^{-1}(y) : y \in \mathbb{R}^n\} = d_T^n(K).$$

Thus the supremum is attained in Corollary 4.3 if $\dim_* = \dim_T$ and $\dim_T K$ is finite. We show that this is true in general.

Theorem 7.2. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . For a generic $f \in C(K, \mathbb{R}^n)$ we have*

$$\max\{\dim_* f^{-1}(y) : y \in \mathbb{R}^n\} = d_*^n(K).$$

Basically we will follow the proof [4, Theorem 4.1]. We need three lemmas, for the next one see [2, Lemma 7.2].

Lemma 7.3. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $x_0 \in K$. Let $K_m \subset K$ be compact sets such that*

- (i) $\dim_T K_m \geq n$ for all $m \in \mathbb{N}^+$ and
- (ii) $\text{diam}(K_m \cup \{x_0\}) \rightarrow 0$ if $m \rightarrow \infty$.

Then for a generic $f \in C(K, \mathbb{R}^n)$ we have $f(x_0) \in f(K_m)$ for infinitely many m .

Lemma 7.4. *Let $n \in \mathbb{N}^+$ and let K be a compact metric space with $\dim_T K \geq n$. Let \dim_* be one of \dim_T , \dim_H , or \dim_P . There exists $x_0 \in K$ such that for every $d < d_*^n(K)$ and $\varepsilon > 0$ there is a compact set $D \subset B(x_0, \varepsilon) \setminus \{x_0\}$ such that $d_*^n(D \cap U) > d$ for all open sets $U \subset K$ intersecting D .*

Proof. If $d_*^n(K) = 0$ then let $x_0 \in K$ be an accumulation point of K and let $D = \{x_1\}$ such that $x_1 \in B(x_0, \varepsilon) \setminus \{x_0\}$. Thus we may assume that $d_*^n(K) > 0$.

First we prove that there is an $x_0 \in K$ such that $d_*^n(U) = d_*^n(K)$ for all open sets $U \subset K$ containing x_0 . Assume to the contrary that for all $x \in K$ there is an open set $U_x \subset K$ such that $x \in U_x$ and $d_*^n(U_x) < d_*^n(K)$. As K is compact, $\{U_x : x \in K\}$ contains a finite cover $\{U_1, \dots, U_k\}$ of K . Then $\bigcup_{i=1}^k U_i = K$ and $\sup\{d_*^n(U_i) : 1 \leq i \leq k\} < d_*^n(K)$, which contradicts the countable stability of d_*^n for F_σ sets.

Now assume that $d < d_*^n(K)$ and $\varepsilon > 0$ are given. Let $B_0 = B(x_0, \varepsilon)$, then clearly $d_*^n(B_0) = d_*^n(K)$. First we prove that there is a compact set C such that $C \subset B_0 \setminus \{x_0\}$ and $d_*^n(C) > d$. Assume to the contrary that there is no such C , then the sets $B_i = B(x_0, \varepsilon) \setminus U(x_0, 1/i)$ satisfy $d_*^n(B_i) \leq d$ for all $i \in \mathbb{N}^+$. Clearly $B_0 = \bigcup_{i=1}^{\infty} B_i \cup \{x_0\}$, and

$$\sup_{i \geq 1} d_*^n(B_i \cup \{x_0\}) \leq \max\{d, 0\} < d_*^n(K) = d_*^n(B_0),$$

which contradicts the countable stability of d_*^n for closed sets.

Finally, let $C \subset B(x_0, \varepsilon) \setminus \{x_0\}$ be a compact set with $d_*^n(C) > d$. We show that there is a compact set $D \subset C$ such that $d_*^n(D \cap U) > d$ for all open sets $U \subset K$ intersecting D . Let \mathcal{U} be a countable open basis for C and define

$$D = C \setminus \bigcup \{U \in \mathcal{U} : d_*^n(U) \leq d\}.$$

Clearly D is compact, and the countable stability of d_*^n for F_σ sets yields that $d_*^n(C \setminus D) \leq d$. Assume to the contrary that there is an $U \in \mathcal{U}$ intersecting D such that $d_*^n(D \cap U) \leq d$. Then clearly $d_*^n(U \setminus D) \leq d_*^n(C \setminus D) \leq d$ and the definition of D yields that $d_*^n(U) > d$. Therefore

$$\max\{d_*^n(D \cap U), d_*^n(U \setminus D)\} \leq d < d_*^n(U),$$

which contradicts the countable stability of d_*^n for F_σ sets. \square

The following lemma will be the heart of the proof.

Lemma 7.5. *Let $n \in \mathbb{N}^+$ and let $D \subset K$ be compact metric spaces with $x_0 \in K \setminus D$. Assume that $0 < d < d_*^n(D \cap U)$ for all open sets $U \subset K$ intersecting D . Then for a generic $f \in C(K, \mathbb{R}^n)$ either $\dim_* f^{-1}(f(x_0)) \geq d$ or $f(x_0) \notin f(D)$.*

Proof. We need to prove that the set

$$\mathcal{F} = \{f \in C(K, \mathbb{R}^n) : \dim_* f^{-1}(f(x_0)) \geq d \text{ or } f(x_0) \notin f(D)\}$$

is co-meager in $C(K, \mathbb{R}^n)$. Consider

$$\Gamma = \{(f, y) \in C(D, \mathbb{R}^n) \times \mathbb{R}^n : \dim_* f^{-1}(y) \geq d \text{ or } y \notin f(D)\}.$$

First assume that Γ is co-meager in $C(D, \mathbb{R}^n) \times \mathbb{R}^n$, we prove that $\mathcal{F} \subset C(K, \mathbb{R}^n)$ is also co-meager. Let

$$R: C(K, \mathbb{R}^n) \rightarrow C(D, \mathbb{R}^n) \times \mathbb{R}^n, \quad R(f) = (f|_D, f(x_0)).$$

Clearly R is continuous, and Tietze's extension theorem implies that it is open. Thus Lemma 2.12 yields that $\mathcal{F} = R^{-1}(\Gamma)$ is co-meager.

Finally, we prove that Γ is co-meager in $C(D, \mathbb{R}^n) \times \mathbb{R}^n$. Lemma 5.7 yields that Γ is in $\sigma(\mathbf{A})$, so it has the Baire property. Hence it is enough to prove by the Kuratowski-Ulam Theorem [14, Theorem 8.41] that for a generic $f \in C(D, \mathbb{R}^n)$ for a generic $y \in \mathbb{R}^n$ we have $(f, y) \in \Gamma$. Let $\{z_i\}_{i \geq 1}$ be a dense set in D and for all $i, j \in \mathbb{N}^+$ define $B_{i,j} = D \cap B(z_i, 1/j)$. For all i, j let

$$\mathcal{G}_{i,j} = \{f \in C(B_{i,j}, \mathbb{R}^n) : \text{there is a non-empty open set } U_f \subset \mathbb{R}^n \text{ such that } \dim_* f^{-1}(y) \geq d \text{ for all } y \in U_f\},$$

and let

$$R_{i,j}: C(D, \mathbb{R}^n) \rightarrow C(B_{i,j}, \mathbb{R}^n), \quad R_{i,j}(f) = f|_{B_{i,j}}.$$

Define

$$\mathcal{G} = \bigcap_{i,j \in \mathbb{N}^+} R_{i,j}^{-1}(\mathcal{G}_{i,j}).$$

The condition of the lemma and the monotonicity of d_*^n yield that $d_*^n(B_{i,j}) > d$, so $\dim_T B_{i,j} \geq n$. Therefore the Main Theorem implies that $\mathcal{G}_{i,j}$ are co-meager in $C(B_{i,j}, \mathbb{R}^n)$. Corollary 2.13 yields that $R_{i,j}^{-1}(\mathcal{G}_{i,j})$ are co-meager in $C(D, \mathbb{R}^n)$, and as a countable intersection of co-meager sets \mathcal{G} is also co-meager in $C(D, \mathbb{R}^n)$. Fix $f \in \mathcal{G}$. It is sufficient to verify that $\Gamma_f = \{y \in \mathbb{R}^n : (f, y) \in \Gamma\}$ is co-meager. Let $V \subset \mathbb{R}^n$ be an arbitrary non-empty open set, it is enough to prove that $\Gamma_f \cap V$ contains a non-empty open set. We may assume that $f(D) \cap V \neq \emptyset$, otherwise $V \subset \Gamma_f$ and we are done. Then there exist $i, j \in \mathbb{N}^+$ such that $B = B_{i,j}$ satisfies $f(B) \subset V$. The definition of \mathcal{G} implies that there is a non-empty open set $U_{f|_B} \subset V$ such that for all $y \in U_{f|_B}$ we have

$$\dim_* f^{-1}(y) \geq \dim_*(f|_B)^{-1}(y) \geq d.$$

Hence $U_{f|_B} \subset \Gamma_f \cap V$, and the proof of the lemma is complete. \square

Now we are able to prove Theorem 7.2.

Proof of Theorem 7.2. By the Main Theorem it is enough to prove that for a generic $f \in C(K, \mathbb{R}^n)$ there exists $y_f \in \mathbb{R}^n$ such that $\dim_* f^{-1}(y_f) \geq d_*^n(K)$. We may assume that $d_*^n(K) > 0$, and let d_m be a positive sequence such that $d_m \nearrow d_*^n(K)$. By Lemma 7.4 there exists $x_0 \in K$ such that for each $m \geq 1$ there is a compact set $K_m \subset B(x_0, 1/m) \setminus \{x_0\}$ such that $d_*^n(K_m \cap U) > d_m$ for all open sets $U \subset K$ intersecting K_m . As $d_*^n(K_m) \geq d_m > 0$, Fact 3.4 yields $\dim_T K_m \geq n$. Therefore we can apply Lemma 7.3 for the sequence $\{K_m\}_{m \geq 1}$ in K , and Lemma 7.5 for all $K_m \subset K$ with d_m . These imply that for a generic $f \in C(K, \mathbb{R}^n)$ we have $f(x_0) \in f(K_m)$ for infinitely many m , and for every m either $d_*^n(f^{-1}(f(x_0))) \geq d_m$ or $f(x_0) \notin f(K_m)$. Hence there is a strictly increasing sequence $\{m_i\}_{i \geq 1}$ depending on f such that $d_*^n(f^{-1}(f(x_0))) \geq d_{m_i}$ for all i , that is,

$$d_*^n(f^{-1}(f(x_0))) \geq \sup\{d_{m_i} : i \geq 1\} = d_*^n(K).$$

This concludes the proof. \square

Assume that \dim_* is one of \dim_H or \dim_P , or $\dim_* = \dim_T$ and $\dim_T K = \infty$. Then we prove that Theorem 7.2 is best possible.

Theorem 7.6. *For each $n \in \mathbb{N}^+$ there is a compact set $K \subset \mathbb{R}^{n+1}$ such that for each $f \in C(K, \mathbb{R}^n)$ there is a $y_f \in \mathbb{R}^n$ such that*

- (i) $d_H^n(K) = 1$ and $d_P^n(K) = n + 1$,
- (ii) $\dim_H f^{-1}(y) < 1$ for a generic $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n \setminus \{y_f\}$,
- (iii) $\dim_P f^{-1}(y) < n + 1$ for every $f \in C(K, \mathbb{R}^n)$ for all $y \in \mathbb{R}^n \setminus \{y_f\}$.

Proof. For each $i \in \mathbb{N}^+$ let $C_i \subset [0, 1/i]$ be compact such that $0 \in C_1$ and

$$\dim_H C_i = \dim_P C_i = 1 - 1/i.$$

Define

$$K = \bigcup_{i=1}^{\infty} K_i, \text{ where } K_i = [0, 1/i]^n \times C_i.$$

Let $\mathbf{0}$ denote the origin of \mathbb{R}^{n+1} , then $\mathbf{0} \in K$ yields that K is compact. Since d_H^n and d_P^n are countably stable for closed sets, Lemmas 3.10 and 3.11 imply that

$$\begin{aligned} d_H^n(K) &= \sup_{i \geq 1} d_H^n(K_i) = \sup_{i \geq 1} \dim_H C_i = 1, \\ d_P^n(K) &= \sup_{i \geq 1} d_P^n(K_i) = \sup_{i \geq 1} \dim_P C_i + n = n + 1. \end{aligned}$$

Therefore (i) holds. For every $f \in C(K, \mathbb{R}^n)$ let $y_f = f(\mathbf{0})$. For all $k \in \mathbb{N}^+$ let $D_k = \bigcup_{i=1}^k K_i$ and define

$$\mathcal{F}_k = \{f \in C(D_k, \mathbb{R}^n) : \dim_H f^{-1}(y) \leq d_H^n(D_k) \text{ for all } y \in \mathbb{R}^n\}.$$

For each $k \in \mathbb{N}^+$ let

$$R_k : C(K, \mathbb{R}^n) \rightarrow C(D_k, \mathbb{R}^n), \quad R_k(f) = f|_{D_k}.$$

Finally, define

$$\mathcal{F} = \bigcap_{k=1}^{\infty} R_k^{-1}(\mathcal{F}_k).$$

Our Main Theorem yields that $\mathcal{F}_k \subset C(D_k, \mathbb{R}^n)$ are co-meager, and Corollary 2.13 implies that $R_k^{-1}(\mathcal{F}_k) \subset C(K, \mathbb{R}^n)$ are co-meager as well. As a countable intersection of co-meager sets, $\mathcal{F} \subset C(K, \mathbb{R}^n)$ is also co-meager. Let $f \in \mathcal{F}$ and $y \in \mathbb{R}^n \setminus \{y_f\}$, it is enough to prove that $\dim_H f^{-1}(y) < 1$. Indeed, there is a $k = k(f, y) \in \mathbb{N}^+$ such that $f^{-1}(y) \subset D_k$. Therefore the definition of \mathcal{F} , the countable stability of d_H^n , and Lemma 3.10 imply that

$$\dim_H f^{-1}(y) \leq d_H^n(D_k) = \sup_{i \leq k} d_H^n(K_i) = \sup_{i \leq k} \dim_H C_i = 1 - 1/k < 1.$$

Thus (ii) holds. The countable stability of packing dimension and Lemma 2.11 imply

$$\dim_P D_k = \sup_{i \leq k} \dim_P K_i = \sup_{i \leq k} \dim_P C_i + n = 1 - 1/k + n < n + 1.$$

Finally, let $f \in C(K, \mathbb{R}^n)$ and $y \in \mathbb{R}^n \setminus \{y_f\}$ be arbitrary. Then there exists a $k = k(f, y) \in \mathbb{N}^+$ such that $f^{-1}(y) \subset D_k$, so

$$\dim_P f^{-1}(y) \leq \dim_P D_k < n + 1.$$

Hence (iii) holds, and the proof is complete. \square

Fact 7.7. *There is a compact metric space K such that $\dim_T K = \infty$, and for each $f \in C(K, \mathbb{R}^n)$ there is a $y_f \in \mathbb{R}^n$ such that $\dim_T f^{-1}(y) < \infty$ for all $y \in \mathbb{R}^n \setminus \{y_f\}$.*

Proof. Let $[0, 1]^\omega$ be the Hilbert cube endowed with a complete metric compatible with the product topology, and let $\mathbf{0} = \{0\}^\omega$. Define $K \subset [0, 1]^\omega$ as

$$K = \bigcup_{i=1}^{\infty} K_i, \text{ where } K_i = [0, 1/i]^i \times \{0\}^\omega.$$

As $\mathbf{0} \in K$, the set K is compact. For all $f \in C(K, \mathbb{R}^n)$ and $k \in \mathbb{N}^+$ let $y_f = f(\mathbf{0})$ and $D_k = \bigcup_{i=1}^k K_i$. Fix $f \in C(K, \mathbb{R}^n)$ and $y \in \mathbb{R}^n \setminus \{y_f\}$. Then there exists a $k = k(f, y) \in \mathbb{N}^+$ such that $f^{-1}(y) \subset D_k$. The countable stability of topological dimension for closed sets yields that

$$\dim_T f^{-1}(y) \leq \dim_T D_k = \sup_{i \leq k} \dim_T K_i = k < \infty. \quad \square$$

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